Visual Computing Exercise 8:

Matrices and Quaternions

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Today

- Transformations & Matrices
- Quaternions
- Exercise 08

Transformations

$$x$$

General	Affine	Linear	Rigid
$T(x)$	$Ax + b$	Ax	$Ax + b$
			$A^{-1} = A^T$
			$det(a) = 1$

Want this to become a linear transformation as well

Transformations

- Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Why? Represent translation using matrix multiplication

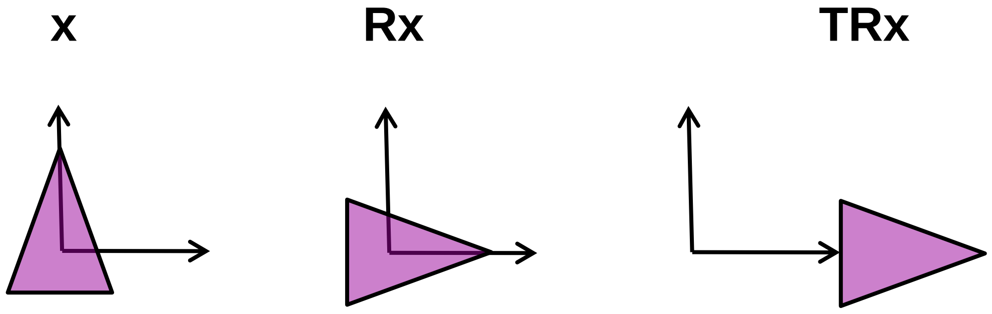
Transformations

- Rotation
- Translation
- Scaling
- Order matters!

$$\begin{bmatrix} \mathbf{R} & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{I} & t \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{S} & 0 \\ 0 & 1 \end{bmatrix}$$

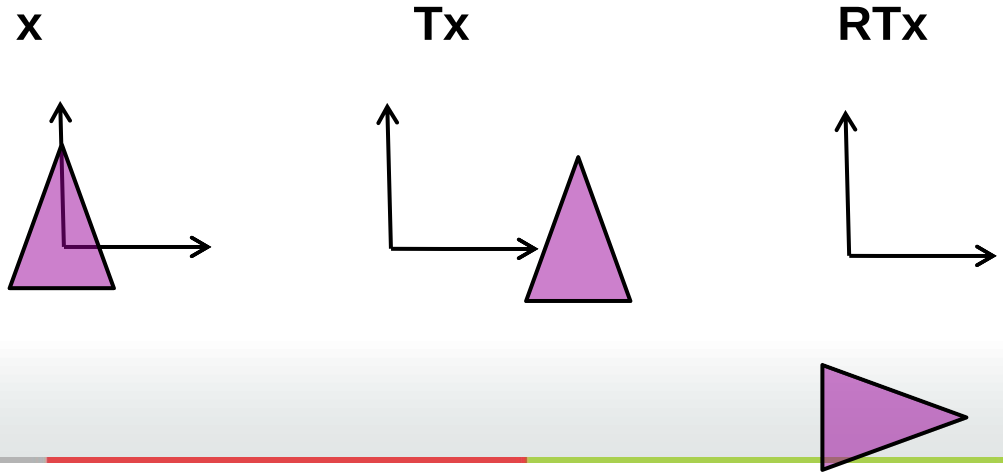
Transformations

- Order matters!
- Rotate and translate



Transformations

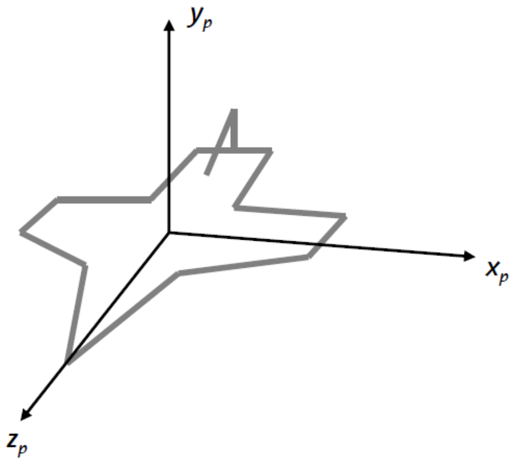
- Order matters!
- Translate and rotate

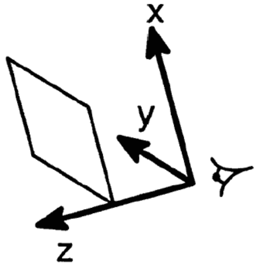


computer graphics laboratory

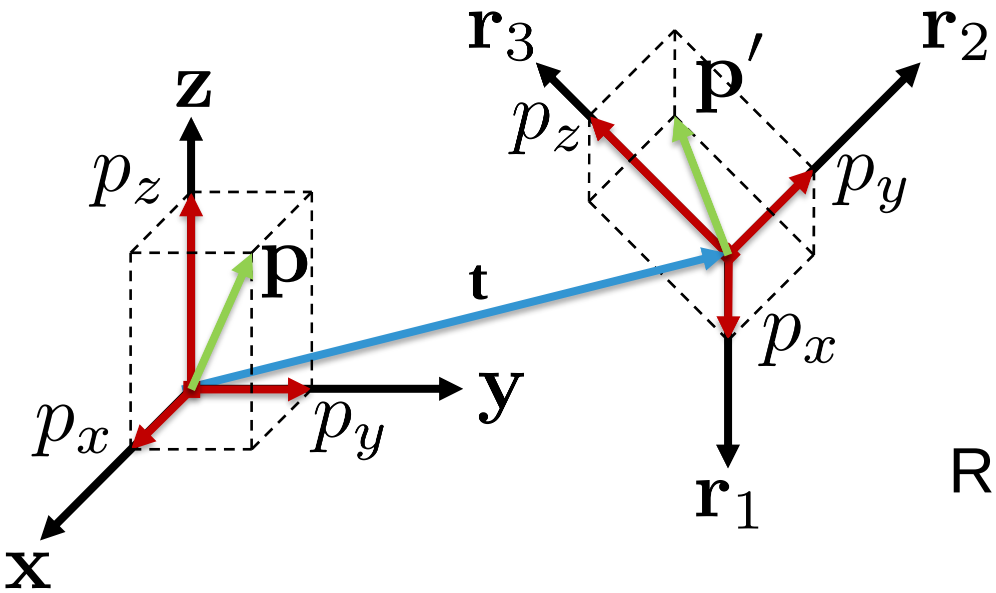
Object to Camera

Example : Object to World to Camera





From the Lecture



$$\mathbf{p}' = \begin{bmatrix} \boxed{\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3} & \boxed{\mathbf{t}} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Rotation Translation

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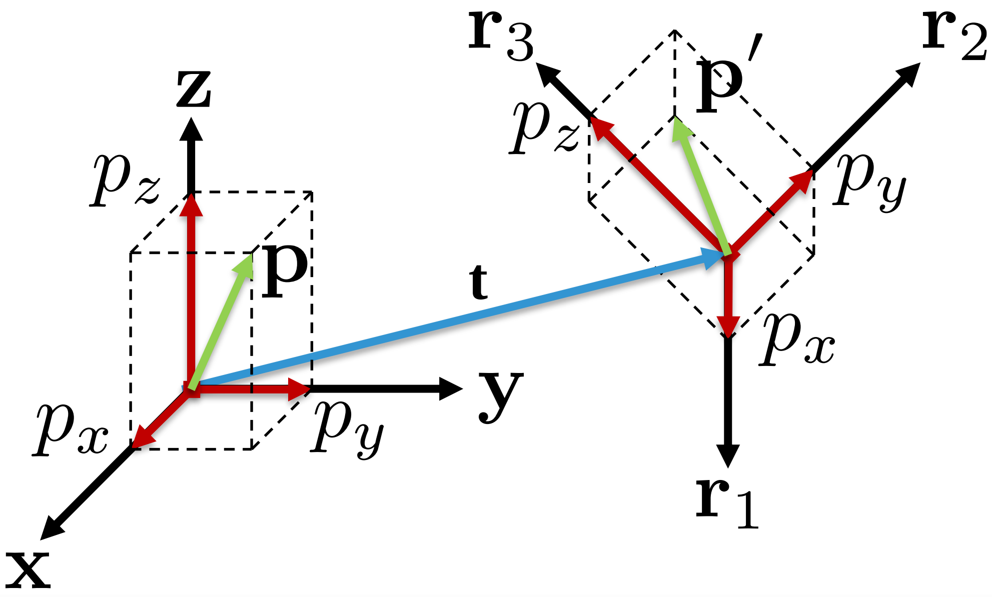
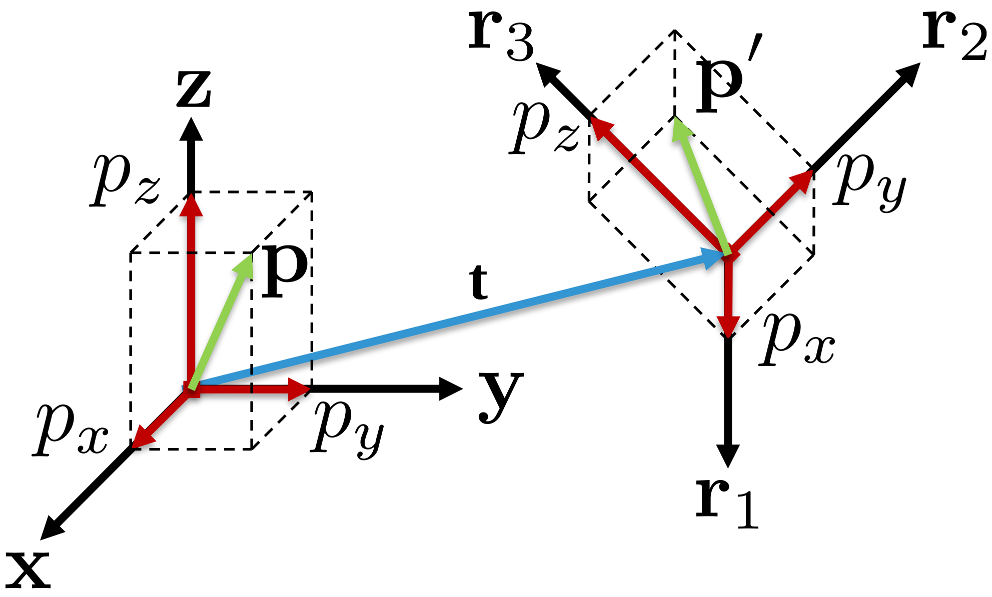
From the Lecture



$$\mathbf{p}' = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{t} + p_x\mathbf{r}_1 + p_y\mathbf{r}_2 + p_z\mathbf{r}_3 \\ 1 \end{bmatrix}$$

From the Lecture



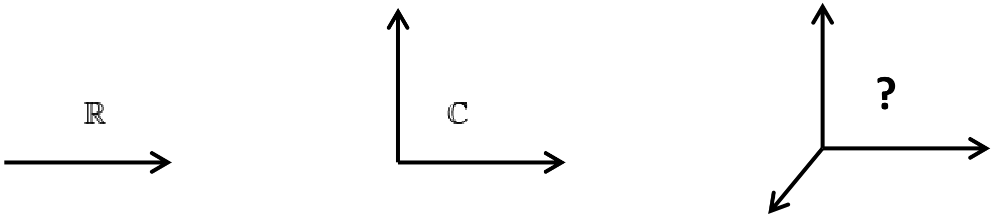
$$\mathbf{p}' = \begin{bmatrix} \boxed{\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3} & \boxed{\mathbf{t}} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$= \mathbf{TRp}$$

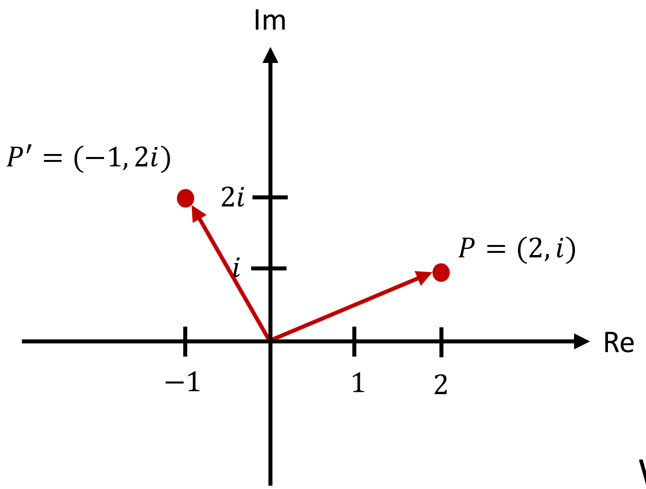
Quaternions

Sir William Rowan Hamilton

Was trying to extend the notion of complex numbers to 3D



Quaternions

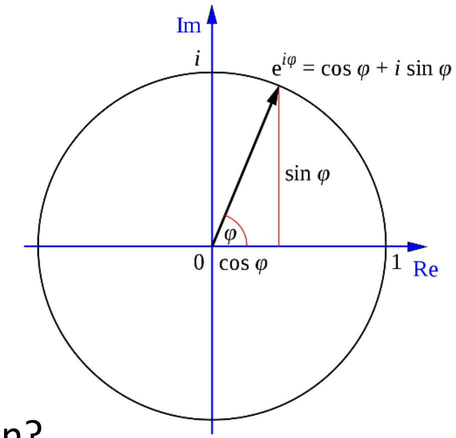


$$P = 2 + i$$

$$P' = P * (0 + i)$$
$$= (2 + i) * (0 + i)$$
$$= 2i + i^2 = 2i - 1$$

$\rightarrow P' = (-1, 2i)$

Euler's Identity



What is the factor for a 45° rotation?

→ (1 + i) (vector length changes!)

Quaternions

Early attempts

1 dimension	a
2 dimensions	$a + ib$

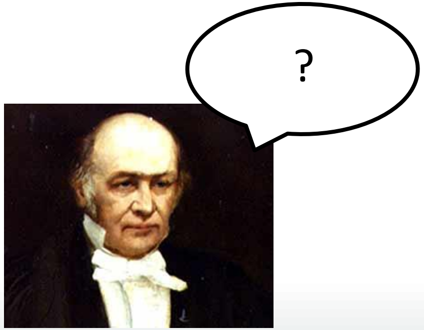
So… 3 dimensions $\quad a + ib + jc \qquad i^2 = j^2 = -1$

Problem

$$\mathbf{z}_1 = a_1 + ib_1 + jc_1$$
$$\mathbf{z}_2 = a_2 + ib_2 + jc_2$$

$$\mathbf{z}_1\mathbf{z}_2 = \blacksquare + i\blacksquare + j\blacksquare + \boxed{ij}\blacksquare + \boxed{ji}\blacksquare$$

$ij = ?$ $\qquad ji = ?$ $\qquad ij = ji?$



Quaternions

Idea : extend the triple into a 4-uple

$$\mathbf{z} = a + ib + jc + kd$$

With

$$i^2 = j^2 = k^2 = -1 \quad \textbf{and} \quad ijk = -1$$

$$\begin{aligned} ij &= k & ji &= -k \\ jk &= i & kj &= -i \\ ki &= j & ik &= -j \end{aligned}$$

$\mathbf{z}$ **is called a quaternion**

Quaternions

A few intuitive properties :

$$a_1 + ib_1 + jc_1 + kd_1 = a_2 + ib_2 + jc_2 + kd_2$$

$$\Longleftrightarrow$$

$$a_1 = a_2 \qquad b_1 = b_2 \qquad c_1 = c_2 \qquad d_1 = d_2$$

Quaternions

A few intuitive properties :

$$\begin{array}{rcccc} & a_1 & + i\, b_1 & + j\, c_1 & + k\, d_1 \\ + & a_2 & + i\, b_2 & + j\, c_2 & + k\, d_2 \\ \hline & (a_1 + a_2) & + i(b_1 + b_2) & + j(c_1 + c_2) & + k(d_1 + d_2) \end{array}$$

Quaternions

A few intuitive properties :

$$\mathbf{q} = a + ib + jc + kd$$

The magnitude is defined by

$$||\mathbf{q}|| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

Unit form of a quaternion

$$\frac{\mathbf{q}}{||\mathbf{q}||}$$

Quaternions

Warning : a quaternion should not be seen as a vector !

$$\mathbf{z} = s + ix + jy + kz = \underbrace{s}_{\text{scalar}} + \underbrace{\mathbf{v}}_{\text{vector}}$$

Quaternions

A quick exercise!

$$\mathbf{z_1} = s_1 + \mathbf{v_1} \qquad\qquad \mathbf{z_2} = s_2 + \mathbf{v_2}$$

Prove that

$$\mathbf{z_1 z_2} = s_1 s_2 - \mathbf{v_1} \cdot \mathbf{v_2} + s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2$$

Hint: Write the quaternions in the form $\mathbf{q} = a + ib + jc + kd$

Quaternions

$$\mathbf{z}_1\mathbf{z}_2 = \boxed{s_1 s_2} - \boxed{\mathbf{v}_1 \cdot \mathbf{v}_2} + \boxed{s_1\mathbf{v}_2} + \boxed{s_2\mathbf{v}_1} + \boxed{\mathbf{v}_1 \times \mathbf{v}_2}$$

= = = = **!** (in general)

$$\mathbf{z}_2\mathbf{z}_1 = \boxed{s_2 s_1} - \boxed{\mathbf{v}_2 \cdot \mathbf{v}_1} + \boxed{s_2\mathbf{v}_1} + \boxed{s_1\mathbf{v}_2} + \boxed{\mathbf{v}_2 \times \mathbf{v}_1}$$

$$\mathbf{z}_2\mathbf{z}_1 \neq \mathbf{z}_1\mathbf{z}_2$$

Quaternions

Conjugate

$$\mathbf{z} = s + \mathbf{v} \qquad \bar{\mathbf{z}} = s - \mathbf{v}$$

Exercise:

Compute $\mathbf{z}\bar{\mathbf{z}} = ||\mathbf{z}||^2$

$$\mathbf{z}^{-1} = \frac{\bar{\mathbf{z}}}{||\mathbf{z}||^2} \qquad 1 = \mathbf{z}\mathbf{z}^{-1} = \mathbf{z}^{-1}\mathbf{z}$$

Quaternions

Fun math, but why talk about it in a CG class?

Point in space $\mathbf{p}$
Rotate by θ **around** $\mathbf{u}$

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\mathbf{R}(\mathbf{u},\theta) = \begin{bmatrix} u_x^2 + \cos\theta(1-u_x^2) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) - u_y\sin\theta & 0 \\ u_x u_y(1-\cos\theta) - u_z\sin\theta & u_y^2 + \cos\theta(1-u_y^2) & u_y u_z(1-\cos\theta) - u_x\sin\theta & 0 \\ u_x u_z(1-\cos\theta) - u_y\sin\theta & u_y u_z(1-\cos\theta) - u_x\sin\theta & u_z^2 + \cos\theta(1-u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Quaternions

Fun math, but why talk about it in a CG class?

New version :

$P = [x, y, z]^T \longrightarrow$ **Quaternion** $\mathbf{p} = 0 + ix + jy + kz$

Rotation : Quaternion $\mathbf{q} = cos(\theta/2) + sin(\theta/2)\mathbf{u}$

$$\boxed{\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}}$$

$$\mathbf{q}^{-1} = cos(\theta/2) - sin(\theta/2)\mathbf{u}$$

Quaternions

$$P = [0, 1, 1]^T \qquad \mathbf{u} = [0, 1, 0]^T \qquad \theta = \pi/2$$

$$\mathbf{p} = 0 + i0 + j + k \qquad \mathbf{q} = cos(\pi/4) + sin(\pi/4)(0 + i0 + j + k0)$$

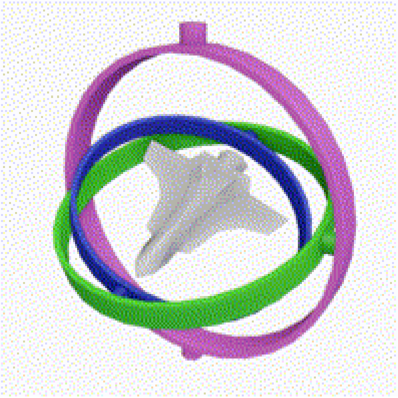
$$= \frac{\sqrt{2}}{2}(1 + j)$$

$$\mathbf{q}^{-1} = \frac{\sqrt{2}}{2}(1 - j)$$

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1} = i + j \qquad \boxed{P' = [1, 1, 0]^T}$$

Quaternions

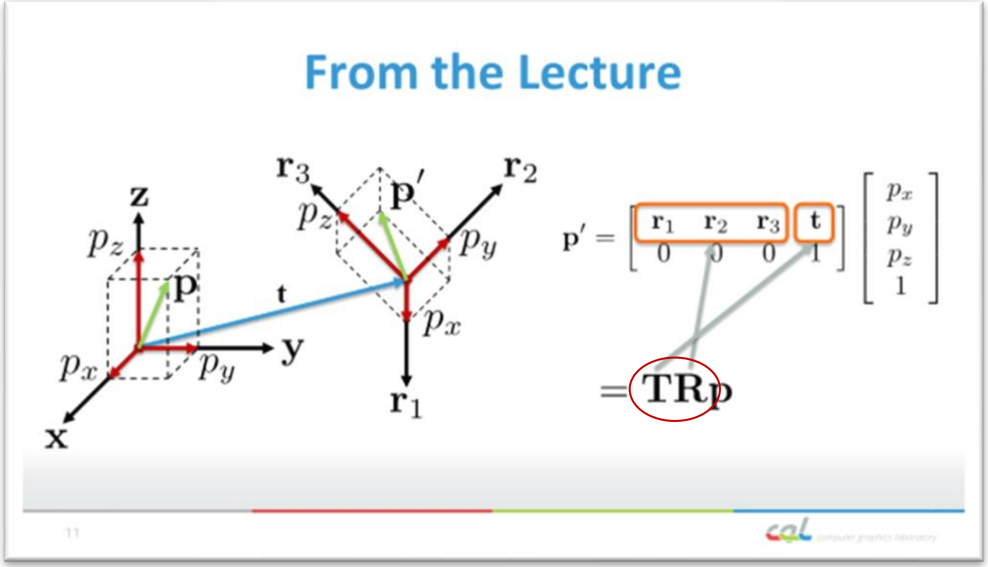
- Why use quaternions?
  - Efficient implementation
  - Easy interpolation
  - No Gimbal lock
- Translation is simple addition
- Rotation representation super easy



Any question so far?

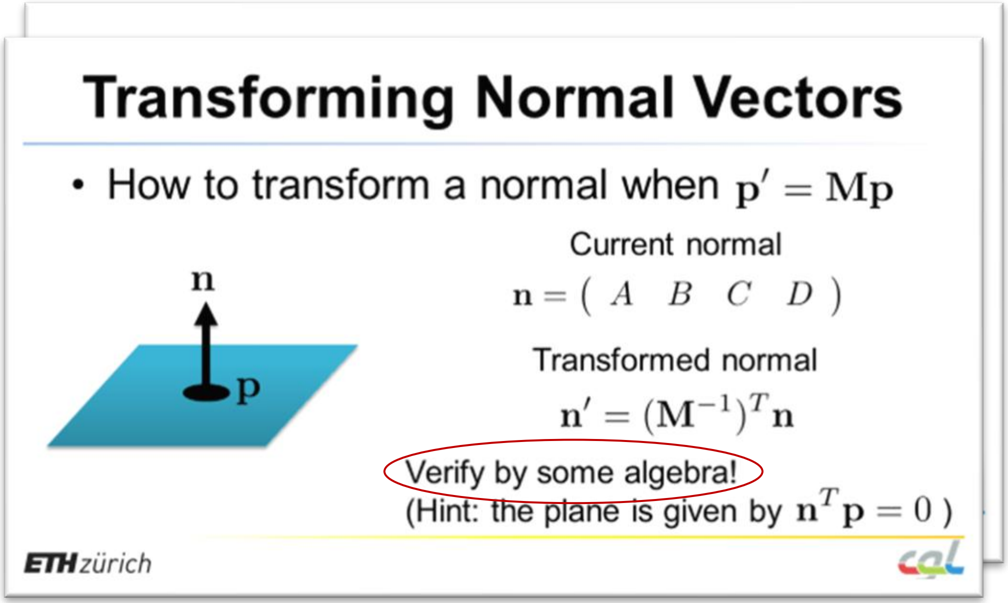
Exercise 08

- Theoretical Part
  - Remember this?



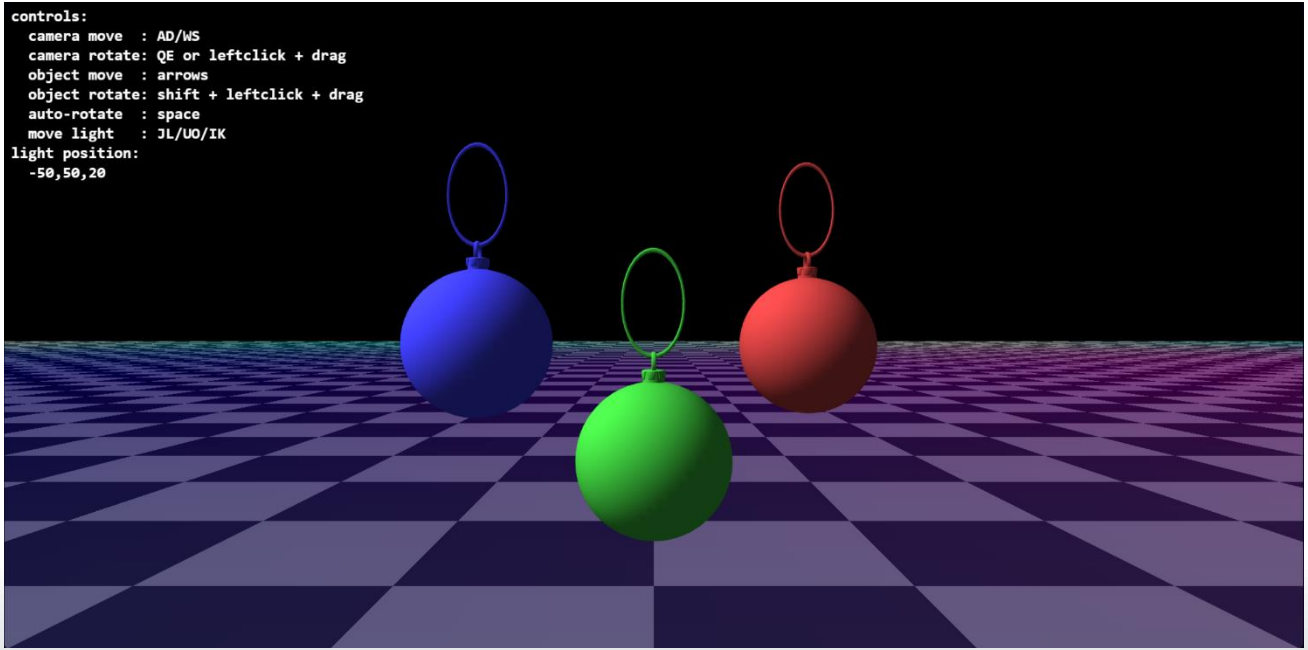
Exercise 08

- Theoretical Part
  - Remember this?
  - And this?



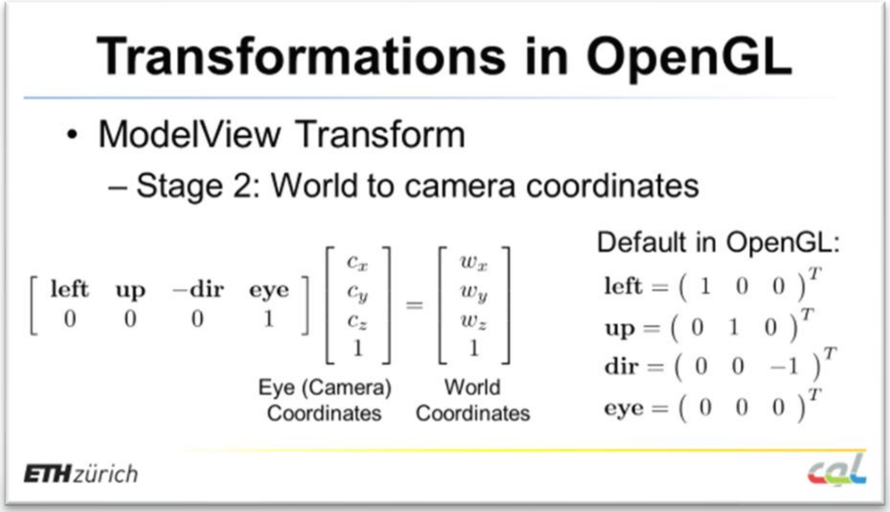
Exercise 08

- Practical Part



Exercise 08

- Practical Part
  - Construct the view matrix and model matrix
  - Add functionality for transformations from key events



Questions?

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