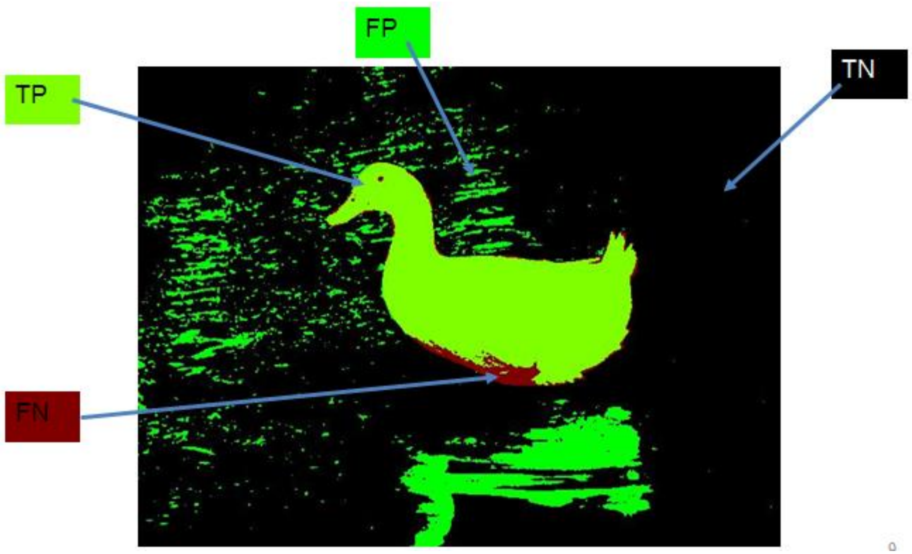
Visual Computing: Convolution and Filtering

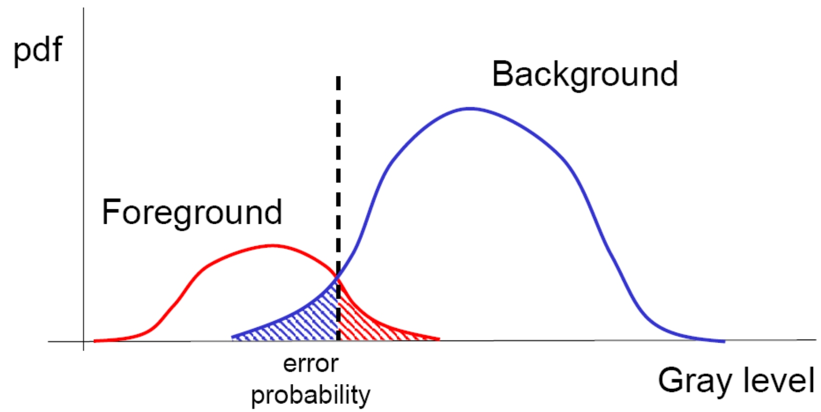
Prof. Marc Pollefeys

ETH

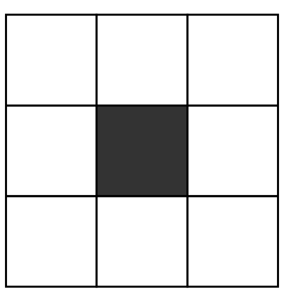
inf | Informatik
Computer Science

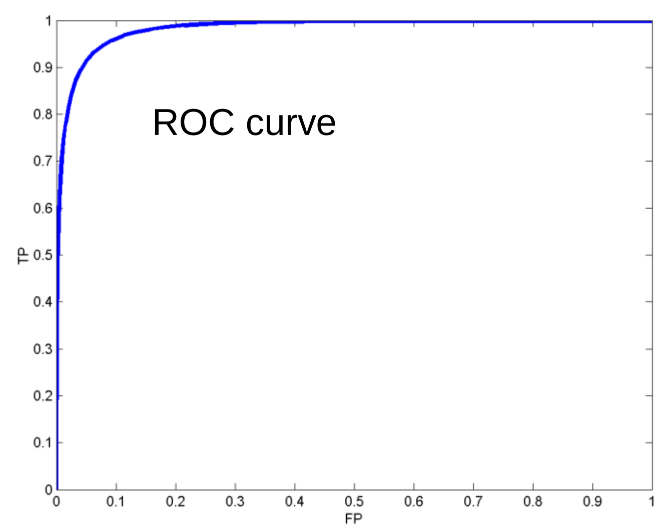
Last time: Segmentation





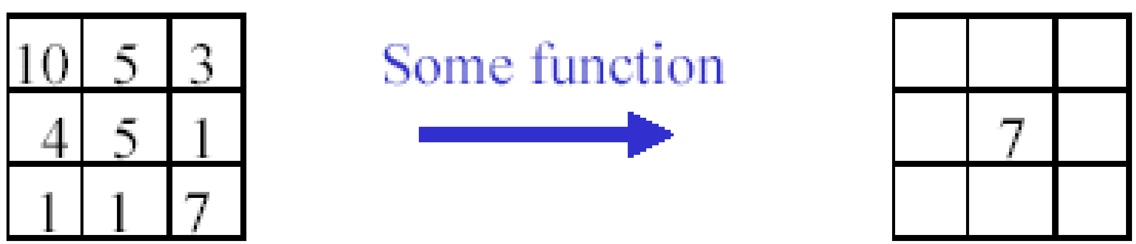
Neighborhood





What is image filtering?

- Modify the pixels in an image based on some function of a local neighborhood of the pixels.



Local image data

Modified image data

Linear Shift-Invariant Filtering

- About modifying pixels based on neighborhood. Local methods simplest.
- Linear means linear combination of neighbors. Linear methods simplest.
- Shift-invariant means doing the same for each pixel. Same for all is simplest.
- Useful to:
  - Low-level image processing operations
  - Smoothing and noise reduction.
  - Sharpen.
  - Detect or enhance features.

Linear Filtering

- *L* is *linear* operation if

$$L\,[\alpha I_1 + \beta I_2] \qquad = \alpha L\,[I_1] + \beta L\,[I_2]$$

Linear Operations: Weighted Sum

- Output *I'* of linear image operation is a weighted sum of each pixel in the input *I*

$$I'_j = \sum_{i=1}^{N} \alpha_{ij} I_i, \; j = 1 \ldots N$$

(note: N=wxh)

Linear Filtering

- Linear operations can be written:

$$I'(x, y) = \sum_{(i,j)\in N(x,y)} K(x, y; i, j) I(i, j)$$

- *I* is the input image; *I’* is the output of the operation.
- K is the *kernel* of the operation. *N*(*m*,*n*) is a neighbourhood of (*m*,*n*).

Linear Filtering

- Linear operations can be written:

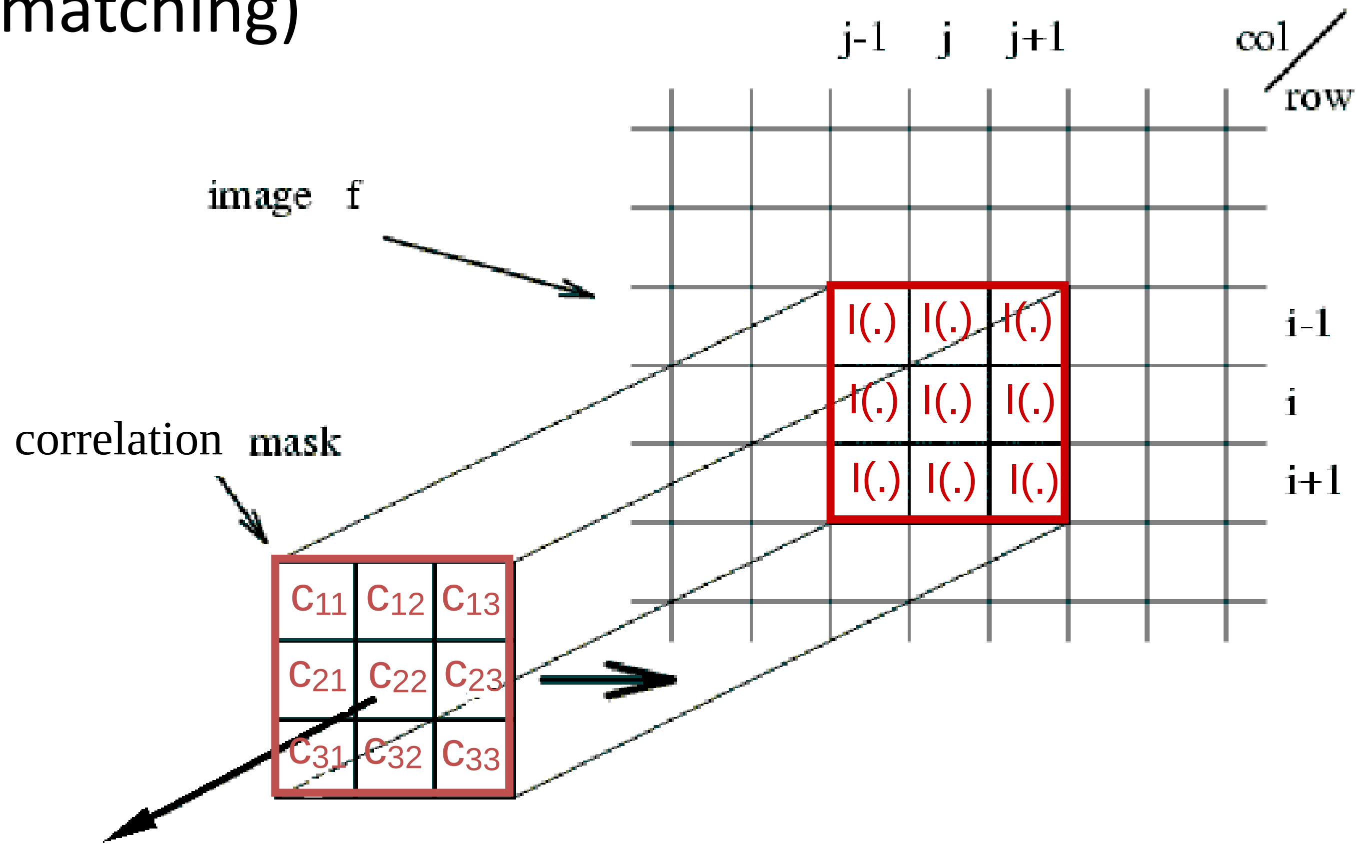
$$I'(x, y) = \sum_{(i,j)\in N(x,y)} K(x, y; i, j) I(i, j)$$

- *I* is the input image; *I'* is the output of the ope
- *K* is n neig

Operations are “shift-invariant” if *K* does NOT depend on (*x*,*y*): using same weights everywhere!

Correlation

(e.g. template matching)



$$
\begin{aligned}
o(i,j) = \; & c_{11}\, I(i-1,j-1) + c_{12}\, I(i-1,j) + c_{13}\, I(i-1,j+1) + \\
& c_{21}\, I(i,j-1) + c_{22}\, I(i,j) + c_{23}\, I(i,j+1) + \\
& c_{31}\, I(i+1,j-1) + c_{32}\, I(i+1,j) + c_{33}\, I(i+1,j+1)
\end{aligned}
$$

Correlation

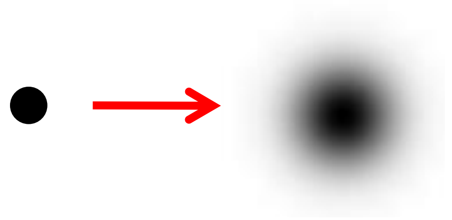
- Linear operation of *correlation*:

$$I' = K \circ I$$

$$I'(x,y) = \sum_{(i,j)\in N(x,y)} K(i,j)I(x+i,y+j)$$

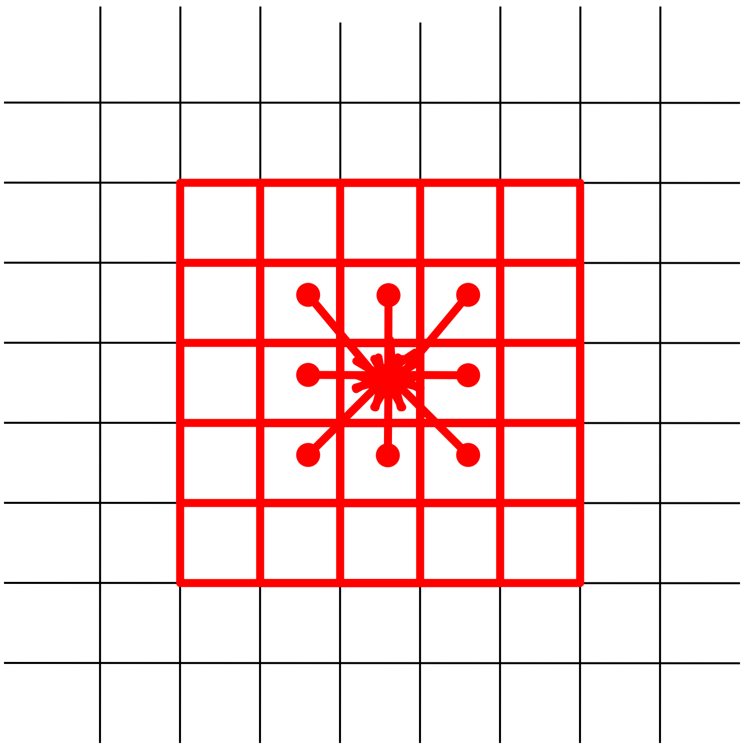
- Represent the linear weights as an image, ***K***

Convolution
(e.g. point spread function)



Kernel

K(-1,-1)	K(0,-1)	K(1,-1)
K(-1,0)	K(0,0)	K(1,0)
K(-1,1)	K(0,1)	K(1,1)



$$\boxed{I'(x,y)} = K(1,1)I(x-1,y-1) + K(0,1)I(x,y-1) + K(-1,1)I(x+1,y-1)$$
$$+ K(1,0)I(x-1,y) + K(0,0)I(x,y) + K(-1,0)I(x+1,y)$$
$$+ K(1,-1)I(x-1,y+1) + K(0,-1)I(x,y+1) + K(-1,-1)I(x+1,y+1)$$

Convolution

- Linear operation of *convolution*:

$$I' = K * I$$

$$I'(x, y) = \sum_{(i,j)\in N(x,y)} K(i, j) I(x - i, y - j)$$

- Represent the linear weights as an image, ***K***
- Same as correlation, but with kernel reversed

ETH

Correlation

$$I'(x, y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(i, j) I(x + i, y + j)$$

Convolution

$$I'(x, y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(i, j) I(x - i, y - j)$$

$$= \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(-i, -j) I(x + i, y + j)$$

So if *K(i,j)* = *K(-i, -j)*, then Correlation == Convolution

Linear Filtering (warm-up)

Original

0	0	0
0	1	0
0	0	0

?

Slide credit: D.A. Forsyth

Linear Filtering (warm-up)

Original

0	0	0
0	1	0
0	0	0

Filtered
(no change)

Slide credit: D.A. Forsyth

Linear Filtering

Original

0	0	0
1	0	0
0	0	0

?

(use convolution)

Slide credit: D.A. Forsyth

Linear Filtering

Original

0	0	0
1	0	0
0	0	0

(use convolution)

Shifted left
By 1 pixel

Slide credit: D.A. Forsyth

Linear Filtering

Original

1	1	1
1	1	1
1	1	1

?

Linear Filtering

Original

1	1	1
1	1	1
1	1	1

Linear Filtering

Original

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

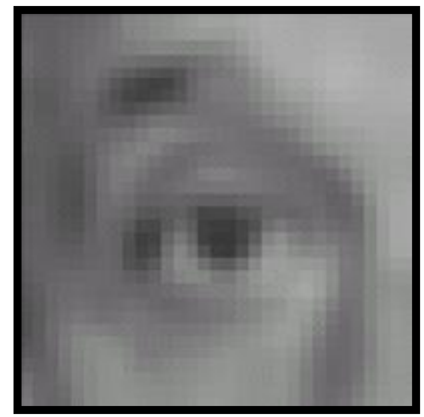
?

Slide credit: D.A. Forsyth

Linear Filtering

Original

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$



Blur (with a box filter)

Slide credit: D.A. Forsyth

Linear Filtering

Original

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 2 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} - \frac{1}{9}\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad ?$$

(Note that filter sums to 1)

Slide credit: D.A. Forsyth

Linear Filtering

Original

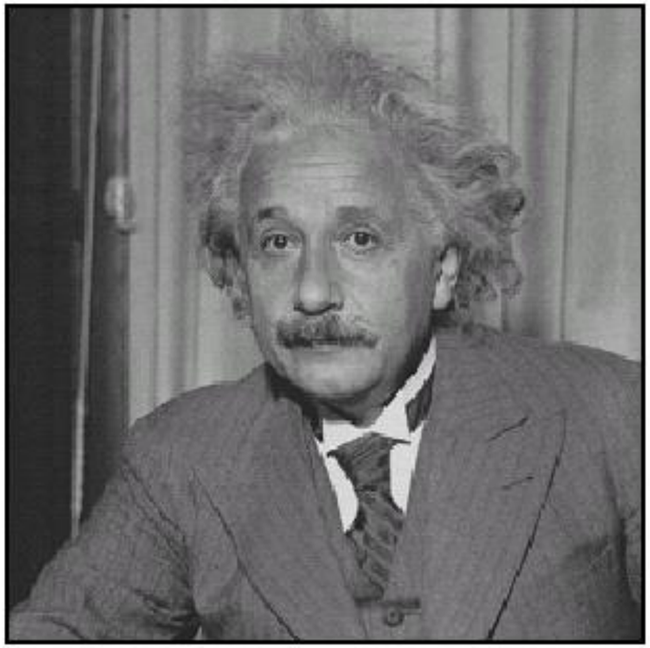
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Sharpening filter

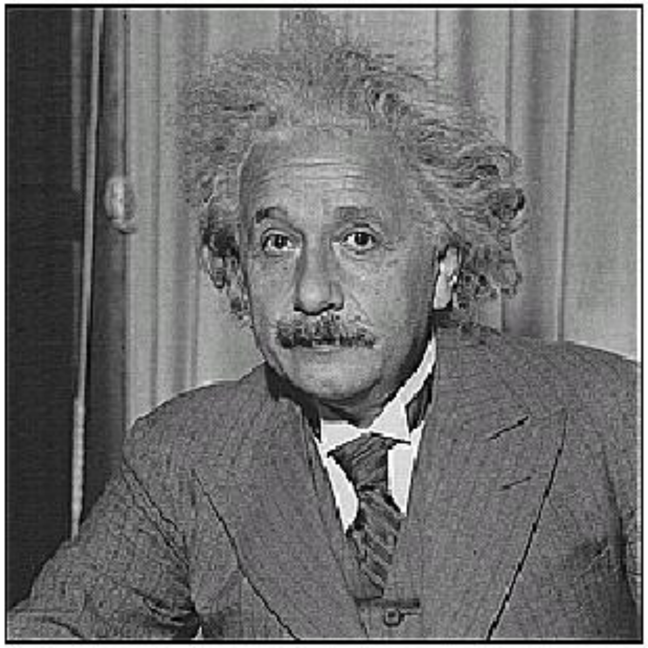
- Accentuates differences with local average

Slide credit: D.A. Forsyth

Sharpening



before

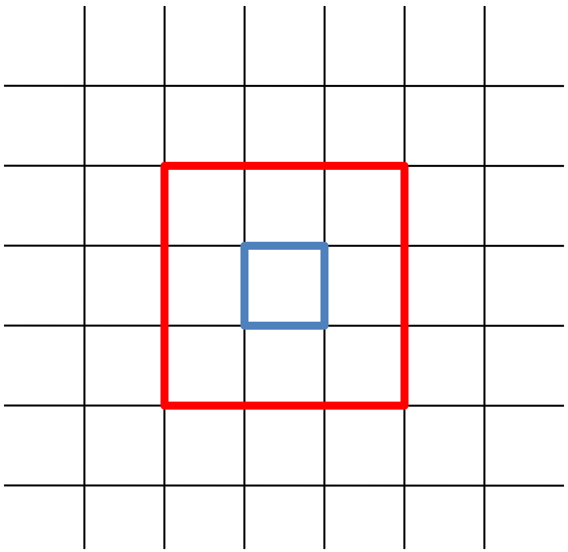


after

Slide credit: D.A. Forsyth

Correlation

(e.g. Template-matching)

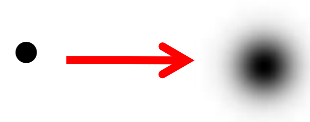


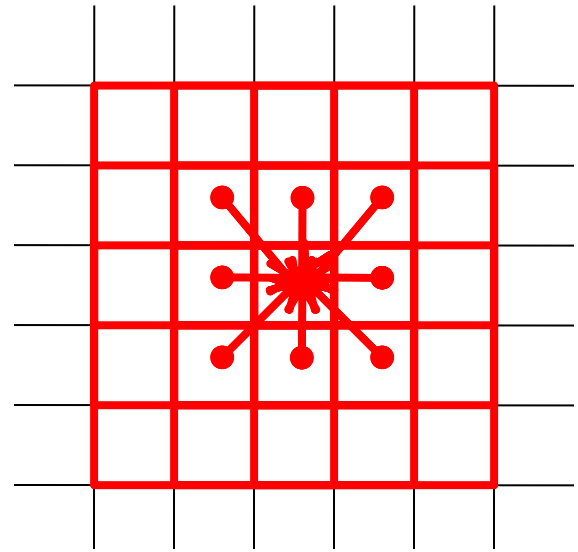
$$I' = \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(i,j) I(x+i, y+j)$$

(matlab default)

Convolution

(e.g. point spread function)





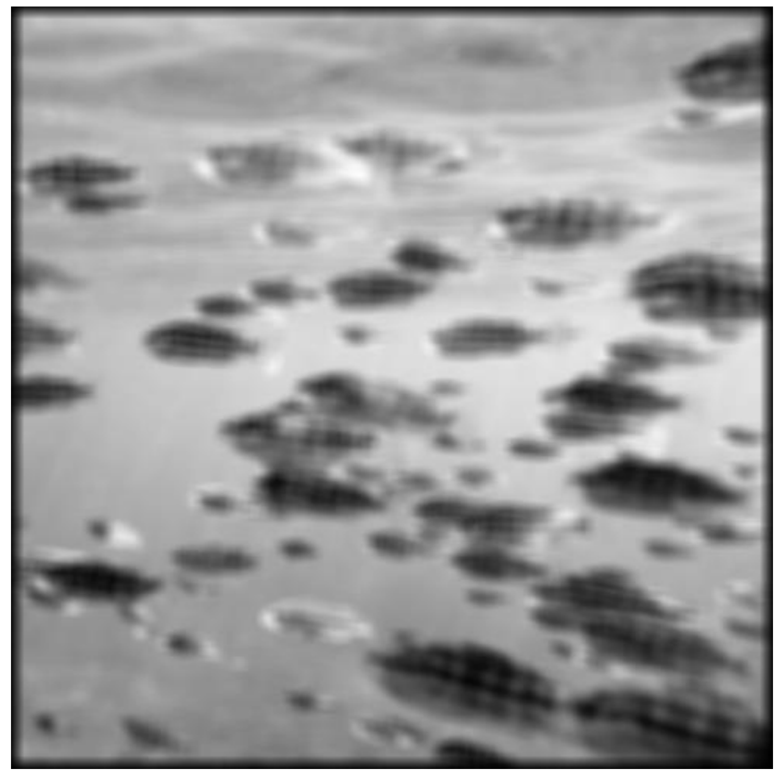
$$I' = \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(i,j) I(x-i, y-j)$$

$$= \sum_{j=-k}^{k} \sum_{i=-k}^{k} K(-i,-j) I(x+i, y+j)$$

Example

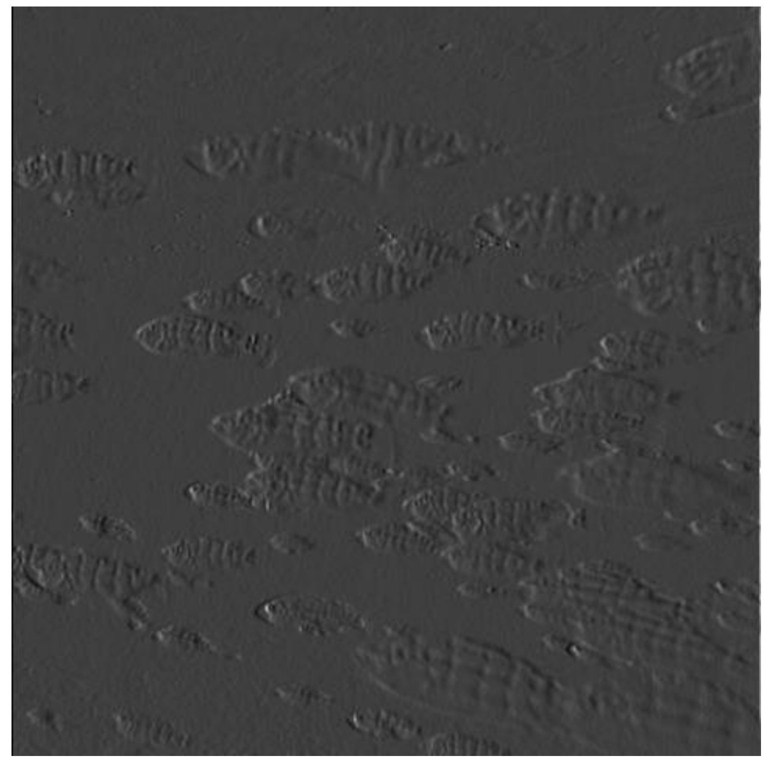
```
K=ones(9,9);

I2=conv2(I,K);
```



Example

$$K = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$



Yucky details

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - copy edge
    - reflect across edge
    - vary filter near edge

Separable Kernels

- Separable filters can be written

  $K(m,n) = f(m)g(n)$

- For a rectangular neighbourhood with size $(2M+1)x(2N+1)$,

$$I'(m,n) = \boxed{f * (\boxed{g * I(N(m,n))})}$$

$$\boxed{I''(m,n)} = \sum_{j=-N}^{N} g(j) I(m, n-j)$$

$$\boxed{I'(m,n)} = \sum_{i=-M}^{M} f(i) I''(m-i, n)$$

Separable Kernels

- Separable filters can be written

  $K(m,n) = f(m)g(n)$

- For a rectangular neighbourhood with size *(2M+1)*x*(2N+1)*,

$$I'(m,n) = \boxed{f * (\boxed{g * I(N(m,n))})}$$

$$\boxed{I''(m,n)} = \sum_{j=-N}^{N} g(j) I(m, n-j)$$

computational advantage?

$$\boxed{I'(m,n)} = \sum_{i=-M}^{M} f(i) I''(m-i, n)$$

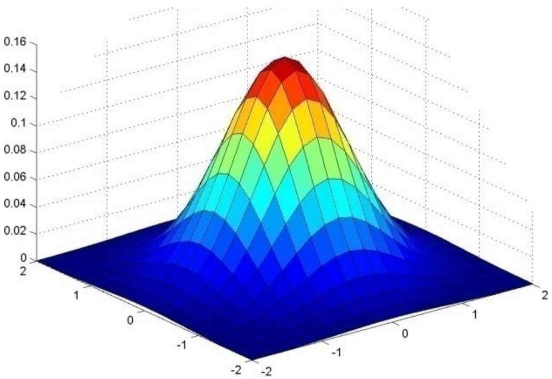
Smoothing Kernels (Low-pass filters)

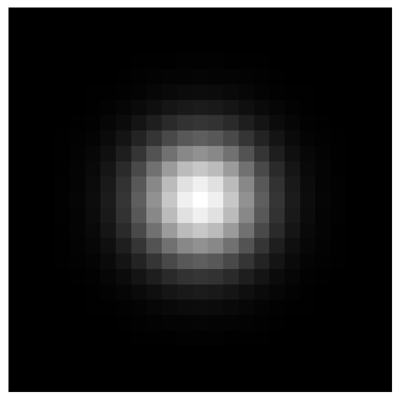
Mean filter: $$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Weighted smoothing filters: $$\frac{1}{10}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Gaussian Kernel

- Idea: Weight contributions of neighboring pixels by nearness





0.003	0.013	0.022	0.013	0.003
0.013	0.059	0.097	0.059	0.013
0.022	0.097	0.159	0.097	0.022
0.013	0.059	0.097	0.059	0.013
0.003	0.013	0.022	0.013	0.003

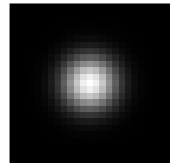
5 x 5, σ = 1

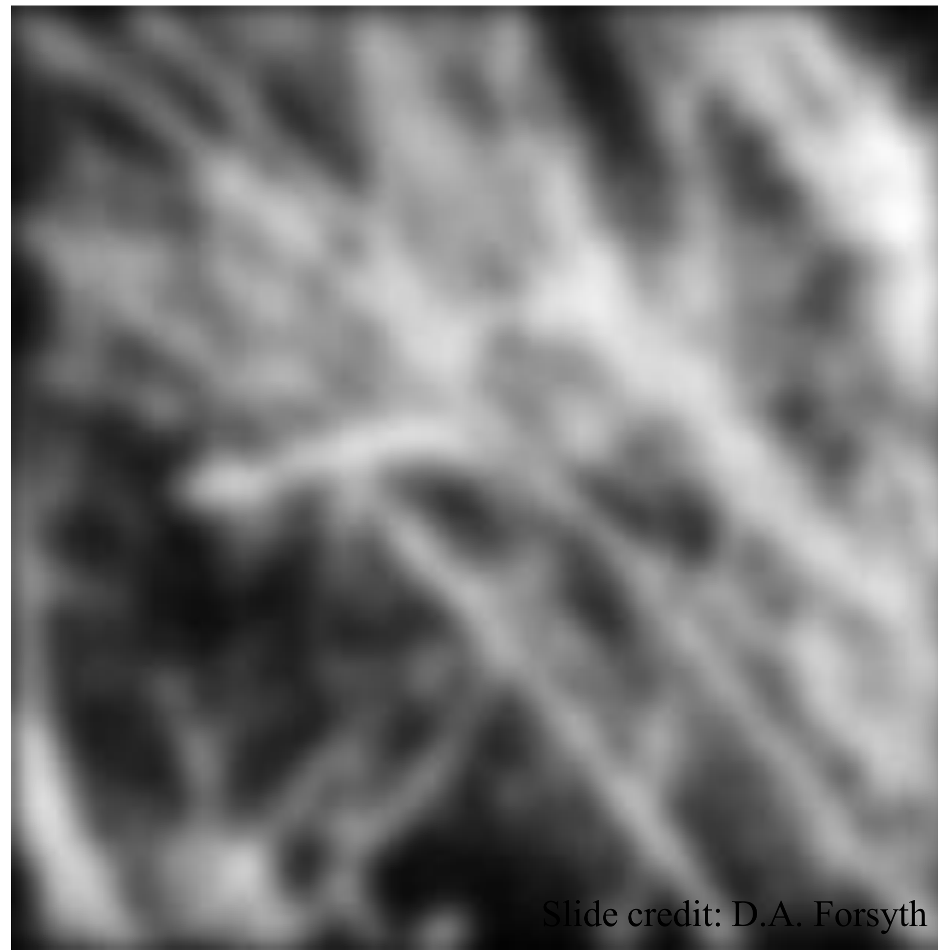
$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

- Constant factor at front makes volume sum to 1

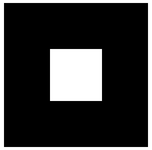
Slide credit: Christopher Rasmussen

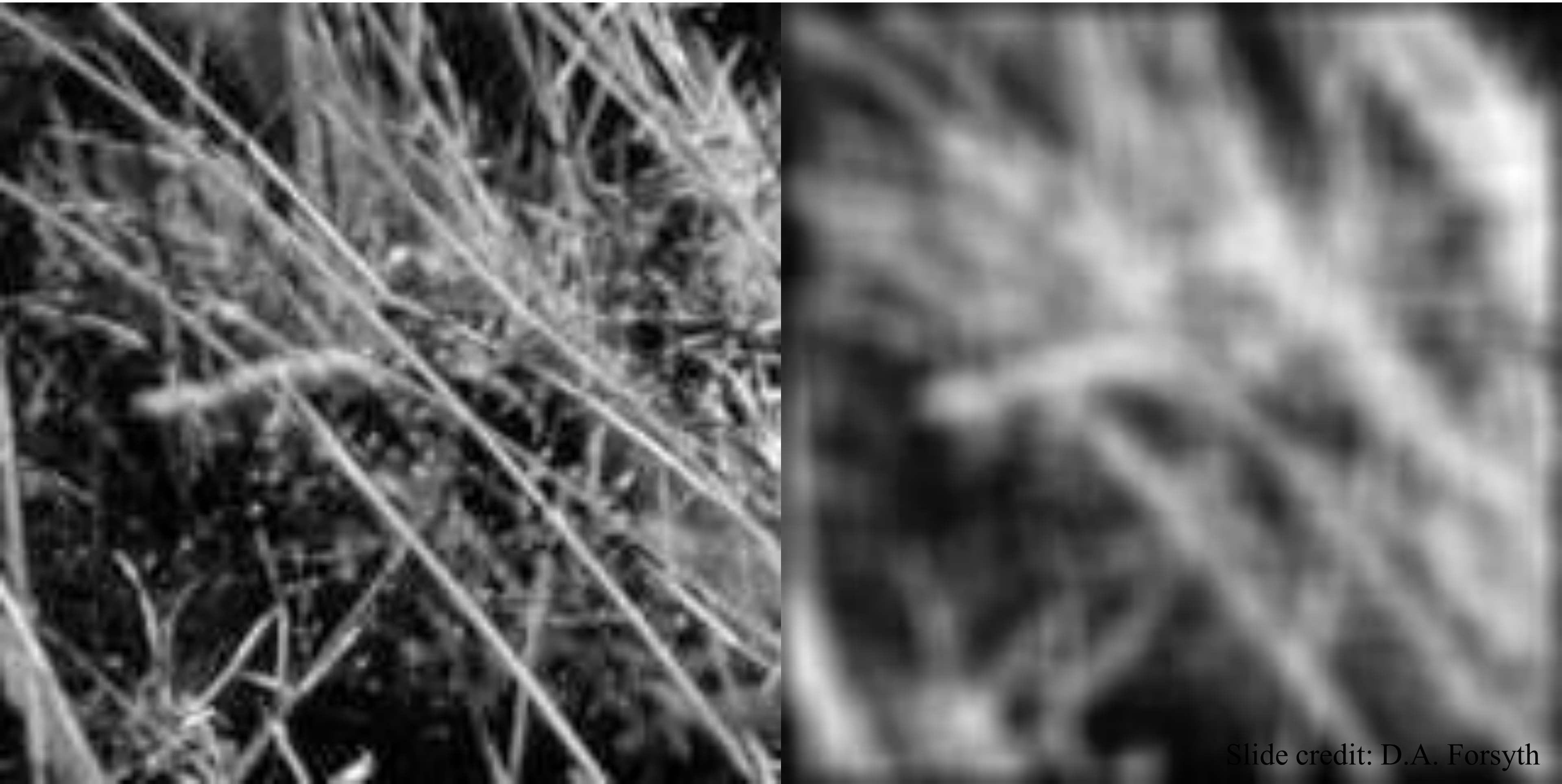
Smoothing with a Gaussian





Smoothing with a box filter





Gaussian Smoothing Kernels

$$g(x, y) = \frac{1}{2\pi\sigma^2} \exp\left[\frac{-\left(x^2 + y^2\right)}{2\sigma^2}\right]$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-x^2}{2\sigma^2}\right] \quad \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-y^2}{2\sigma^2}\right]$$

$$= g(x)g(y)$$

Separable!

Gaussian Smoothing Kernels

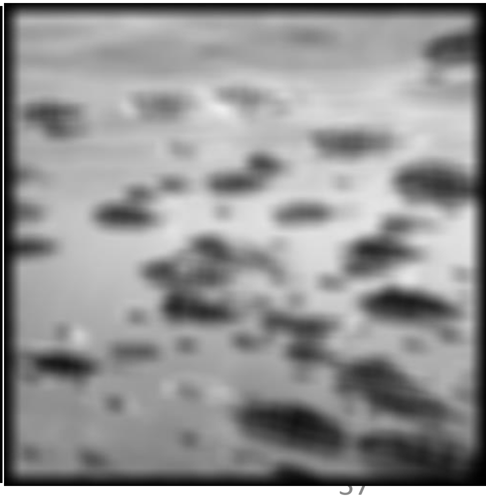
- Amount of smoothing depends on σ and window size.
- Width > 3σ

7x7; σ=1.

7x7; σ=9.

19x19; σ=1.

19x19; σ=9.



Scale Space

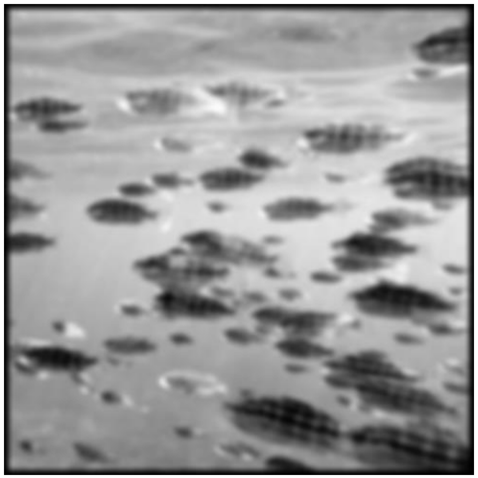
- Convolution of a Gaussian with standard deviation σ with itself is a Gaussian standard deviation $\sigma\sqrt{2}$.

- Repeated convolution by a Gaussian filter produces the scale space of an image.

Scale Space Example

11x11; σ=3.

1



2

3

4

5

6

Gaussian Smoothing Kernel Top-5

1. Rotationally symmetric
2. Has a single lobe

   - Neighbor's influence decreases monotonically
3. Still one lobe in frequency domain

   - No corruption from high frequencies
4. Simple relationship to σ
5. Easy to implement efficiently

Differential Filters

Prewitt operator:

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Sobel operator:

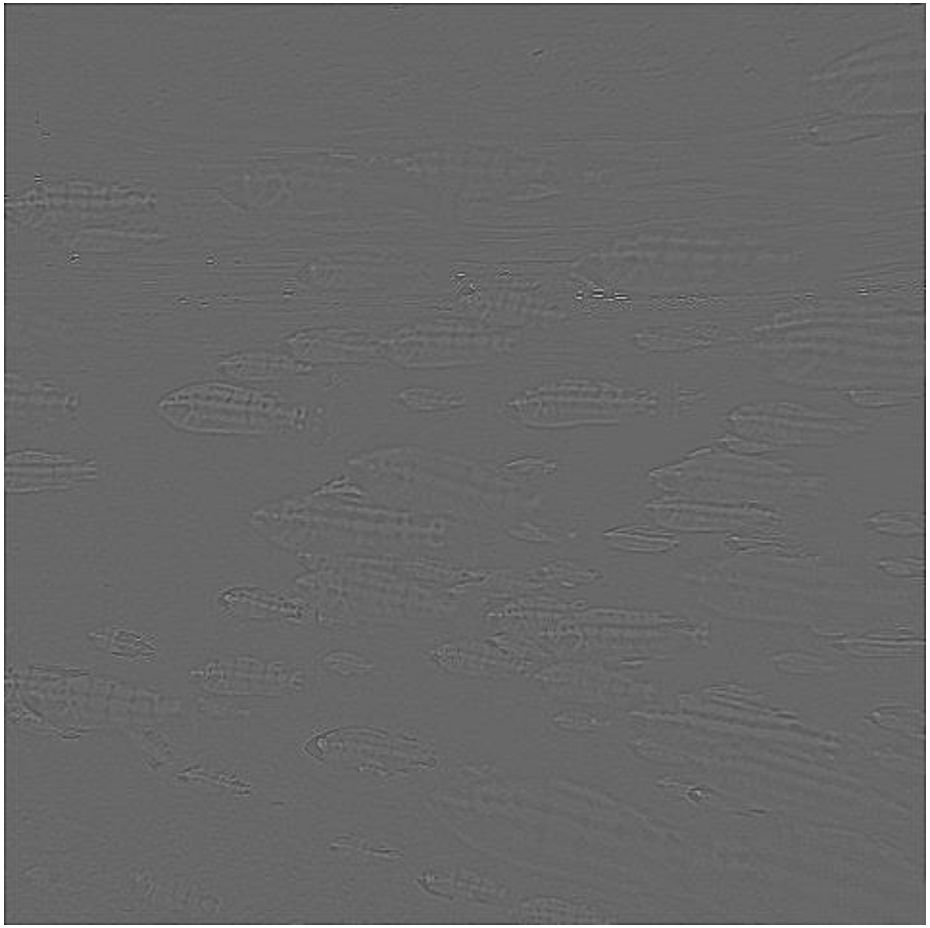
$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

High-pass filters

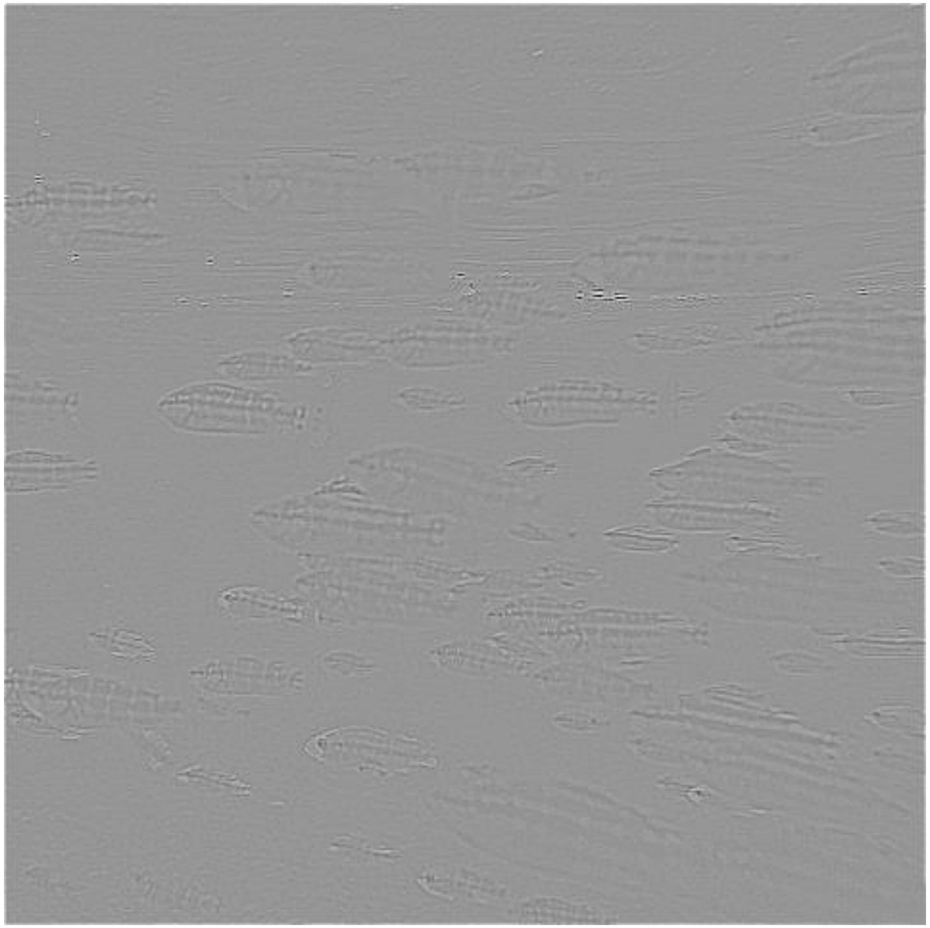
Laplacian operator: $$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

High-pass filter: $$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

High-pass filters



Laplacian



High pass

Differentiation and convolution

- Recall, for 2D function, f(x,y):

$$\frac{\partial f}{\partial x} = \lim_{\varepsilon \to 0}\left(\frac{f(x+\varepsilon, y)}{\varepsilon} - \frac{f(x,y)}{\varepsilon}\right)$$

- This is linear and shift invariant, so must be the result of a convolution.

- We could approximate this as

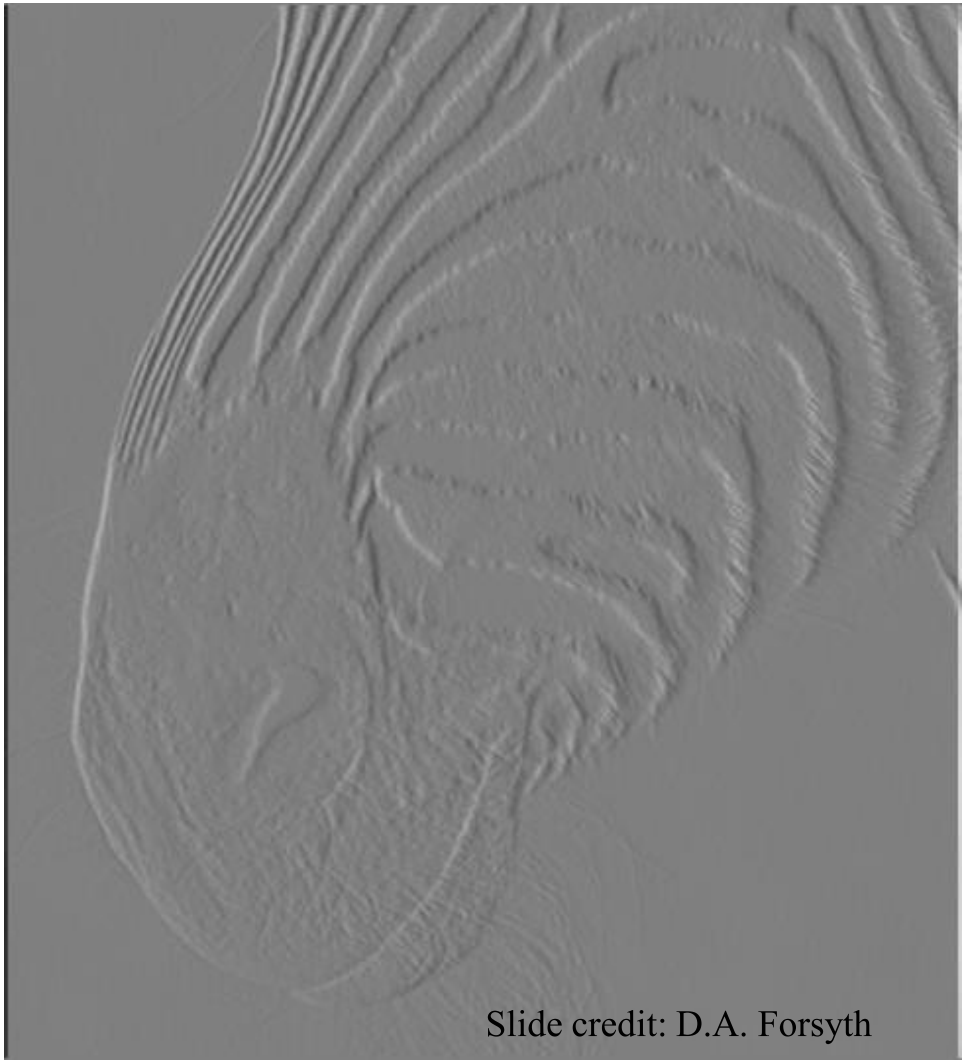
$$\frac{\partial f}{\partial x} \approx \frac{f(x_{n+1}, y) - f(x_n, y)}{\Delta x}$$

(which is obviously a convolution)

-1	**1**

Slide credit: D.A. Forsyth

Vertical gradients from finite differences

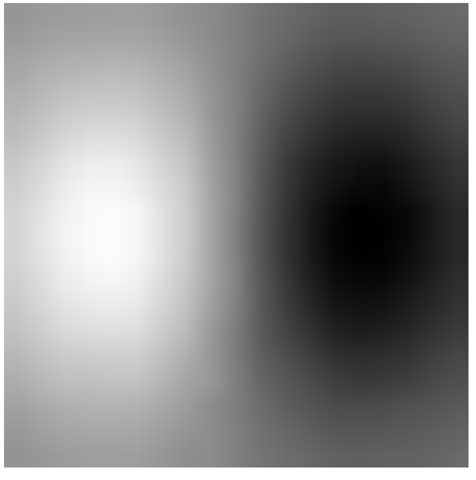


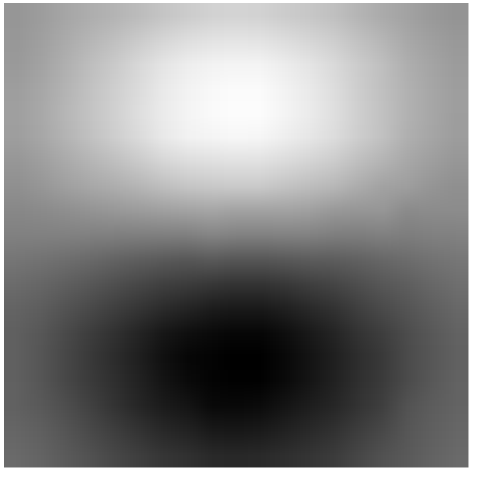
Slide credit: D.A. Forsyth

Filters are templates

- Filter at some point can be seen as taking a dot-product between the image and some vector
- Filtering the image is a set of dot products

- filters look like the effects they are intended to find
- filters find effects they look like





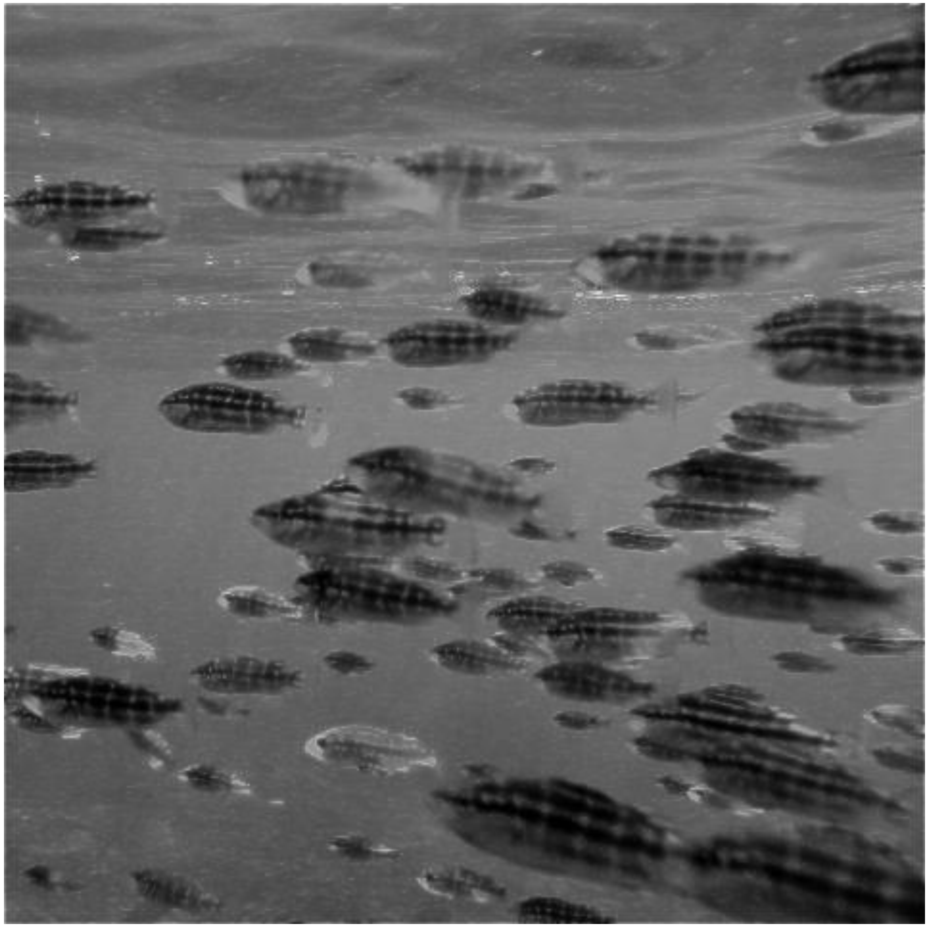
Slide credit: D.A. Forsyth

Image Sharpening

- Also known as Enhancement
- Increases the high frequency components to enhance edges.
- $I' = I + \alpha|k*I|$, where k is a high-pass filter kernel and α is a scalar in [0,1].

Sharpening Example

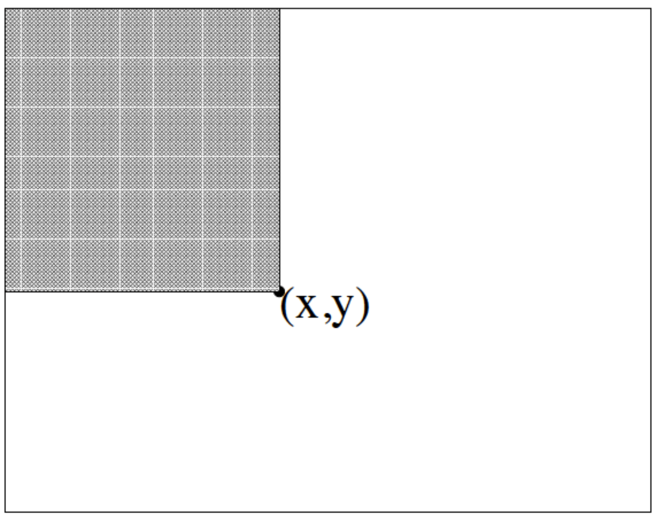
original

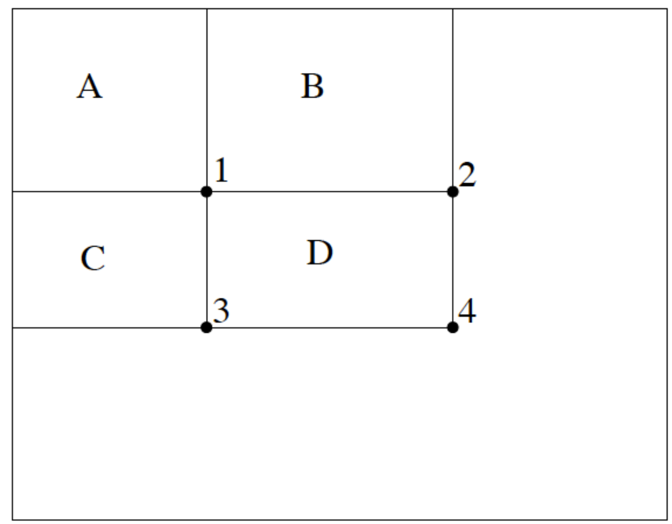


α =0.5

Integral images

- Integral images (also know as summed-area tables) allow to efficiently compute the convolution with a constant rectangle





$$II(x,y)= \int_0^x \int_0^y I(x',y')\,dx'\,dy'$$

A=II(1)
A+B=II(2)

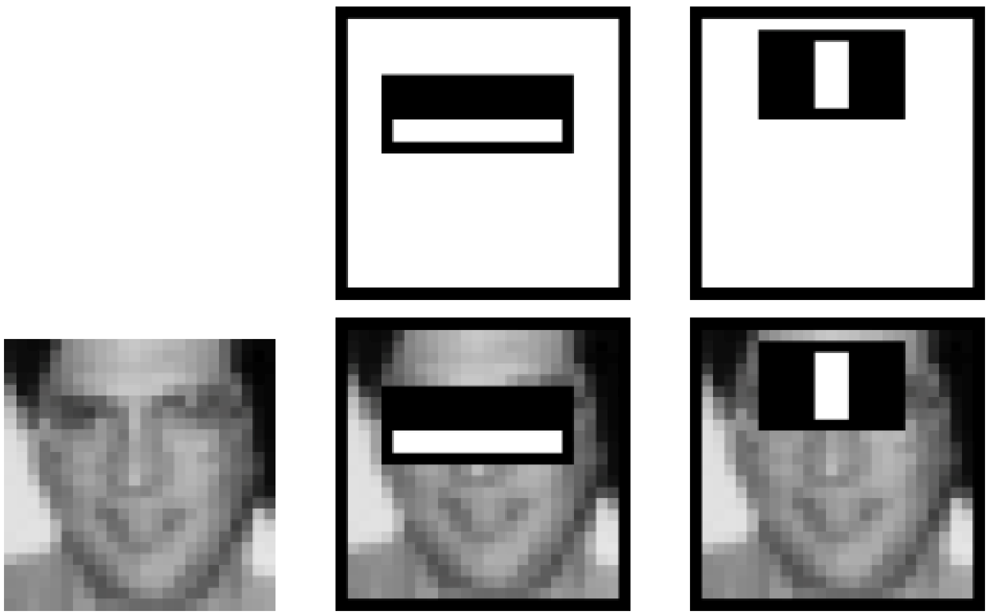
A+C=II(3)
A+B+C+D=II(4)

$\boxed{\text{D=II(4)-II(2)-II(3)+II(1)}}$

Figures from Viola and Jones 2001

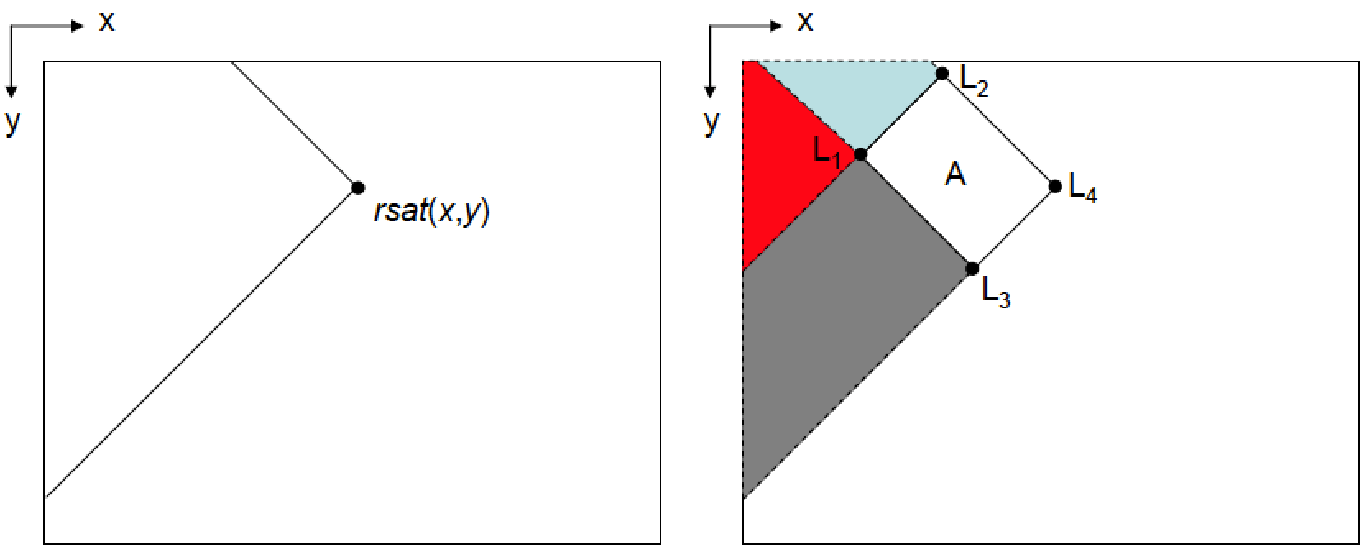
Viola-Jones cascade face detection

- Very efficient face detection using integral images

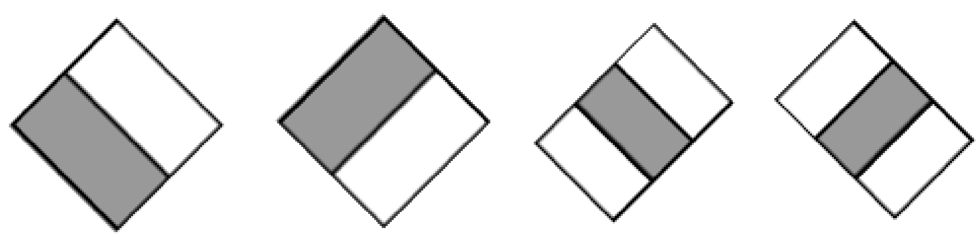


Figures from Viola and Jones 2001

- Also possible along diagonal



(a) (b)



Thursday: Image Features