Visual Computing:
Radon Transform

Marc Pollefeys

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Computer Science

Last lecture: Optical Flow

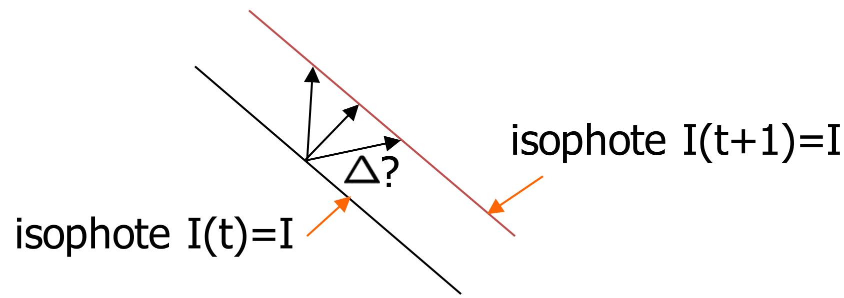
- Brightness constancy equation

$$I_x u + I_y v + I_t = 0$$

- Aperture problem

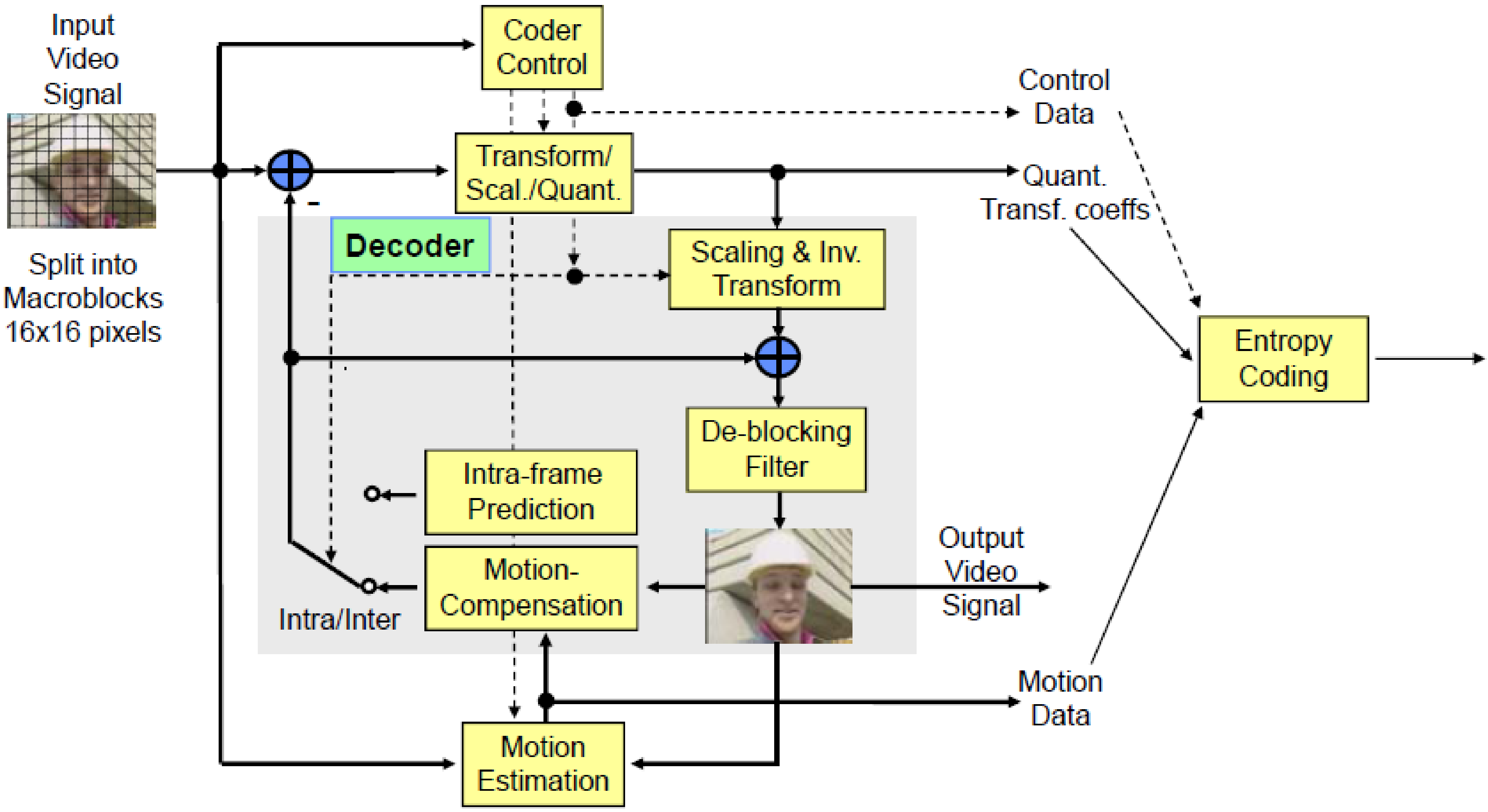
$I_x u + I_y v + I_t = 0$ (1 constraint)

u, v (2 unknowns)

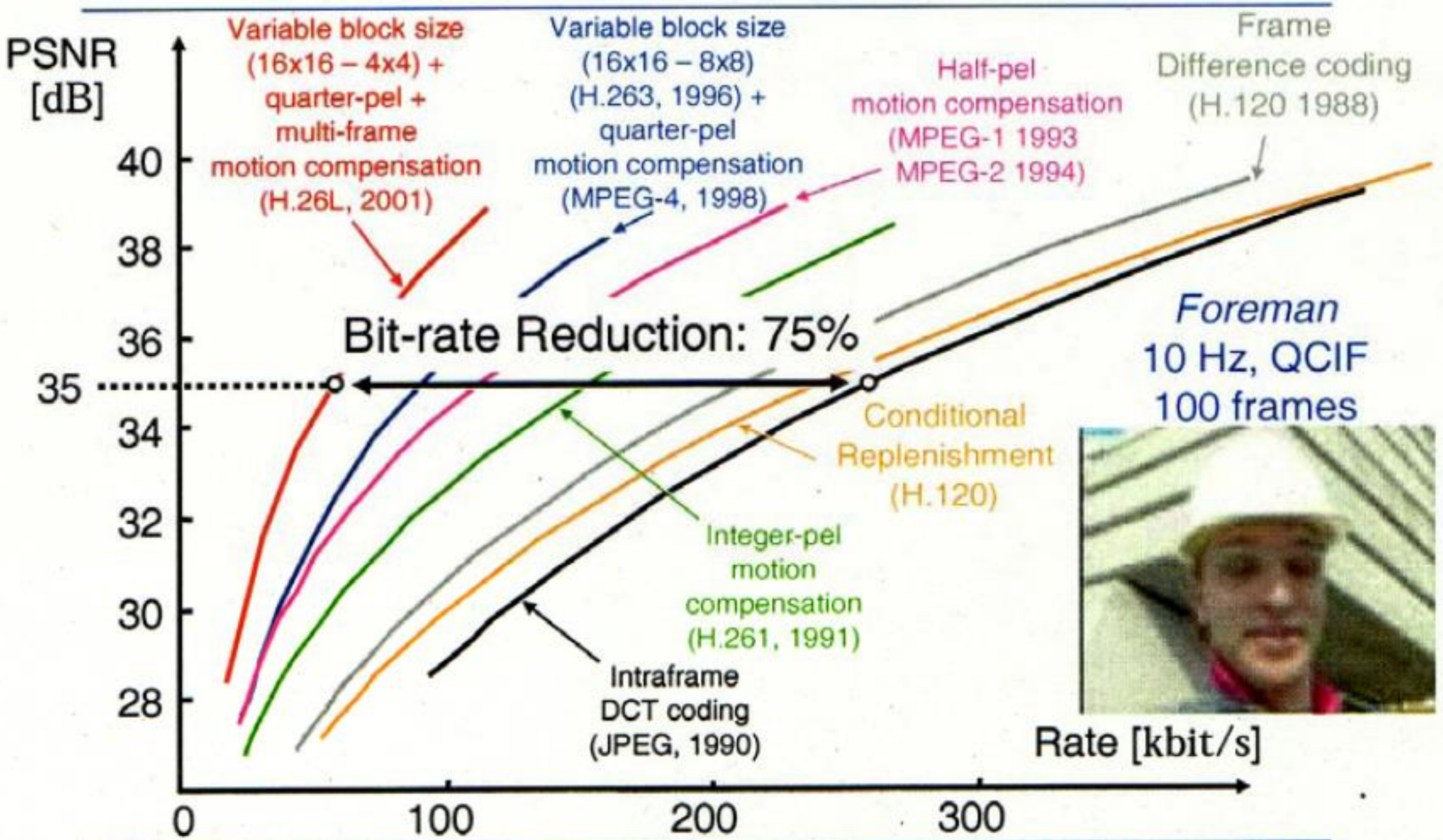


- Solution:
  - regularize (trade-off brightness constancy and smoothness)
  - e.g. Kanade-Lukas-Tomasi $E(\mathbf{h}) = \sum_{\mathbf{x} \in R} [I(\mathbf{x} + \mathbf{h}) - I_0(\mathbf{x})]^2$

MPEG-4 part 10 aka H.264



Milestones in Video Coding



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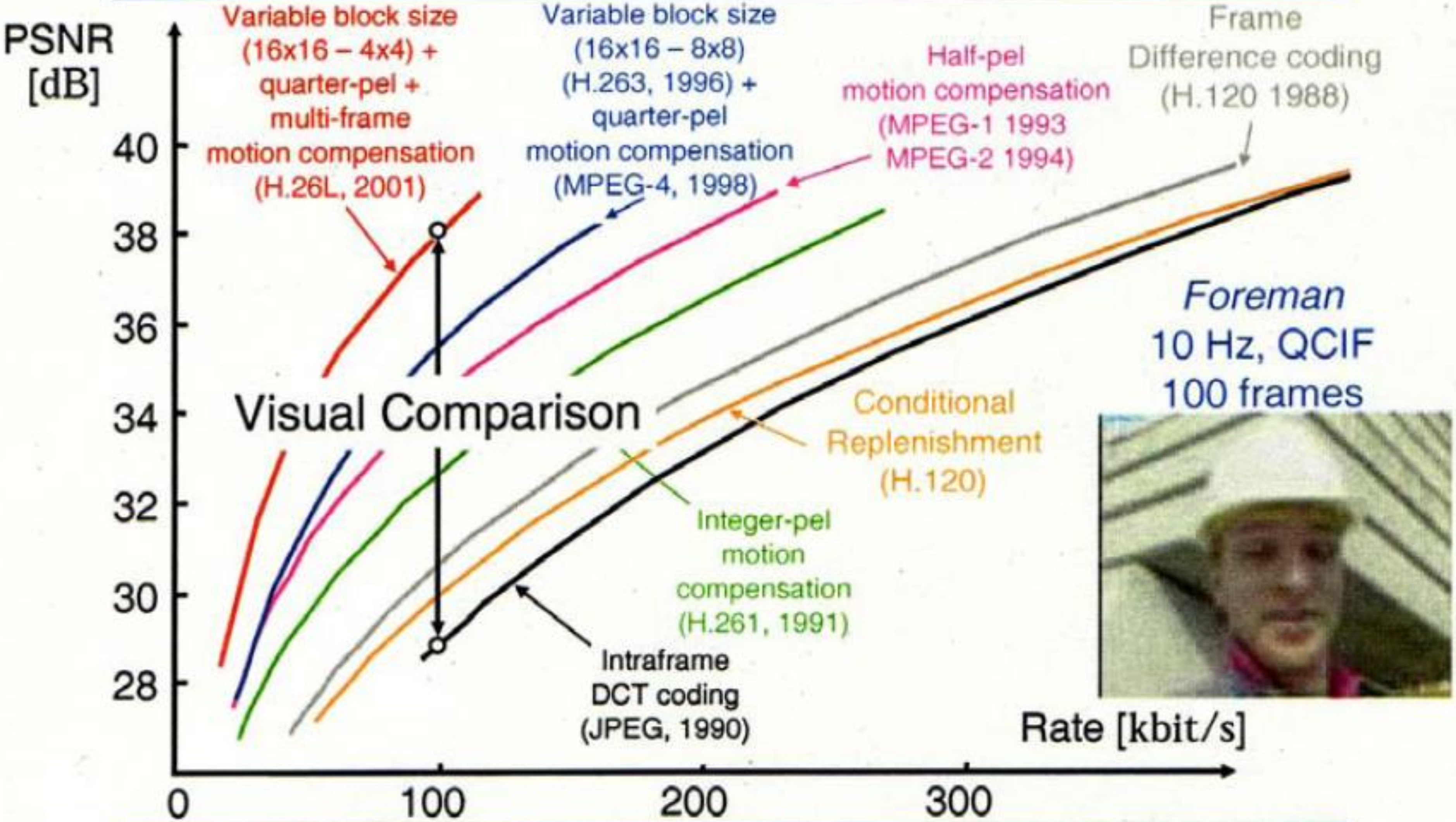
Milestones in Video Coding



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video

Medical Imaging

- 2 forms - radiation source:
  - outside the body: X-ray, ultrasound
  - inside the body: magnetic resonance imaging (MRI), positron emission tomography (PET), single photon emission computed tomography (SPECT)

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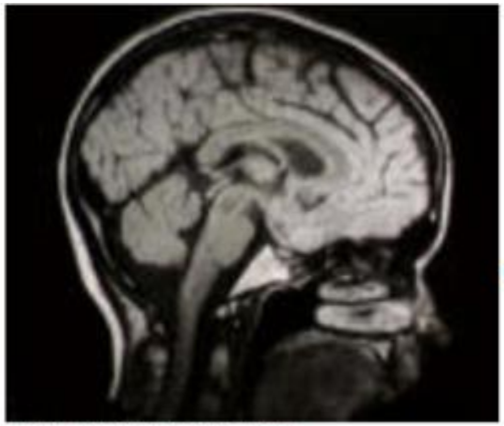
Medical Imaging

- 2 forms - radiation source:
  - outside the body: X-ray, ultrasound
  - inside the body: magnetic resonance imaging (MRI), positron emission tomography (PET), single photon emission computed tomography (SPECT)

- Computed Tomography (CT)
  - 1917 - mathematical basis - Johann Radon
  - 1960s - Cormack & Hounsfield - 1st scanning device

    -> Nobel prize



A CAT scan of the inside of a head.

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Motivation

Computed
Tomography

[tomography <- Greek work *tomos* meaning "cut" or "slice"]

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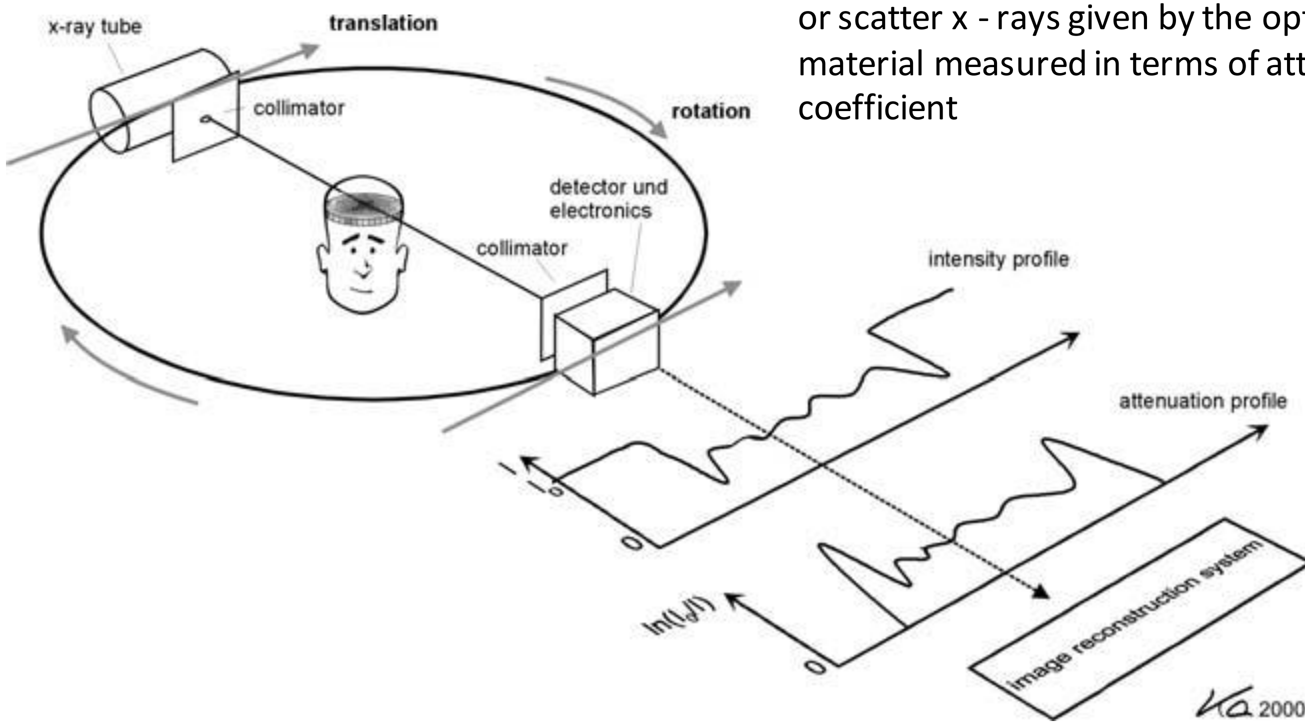
CT: data collection

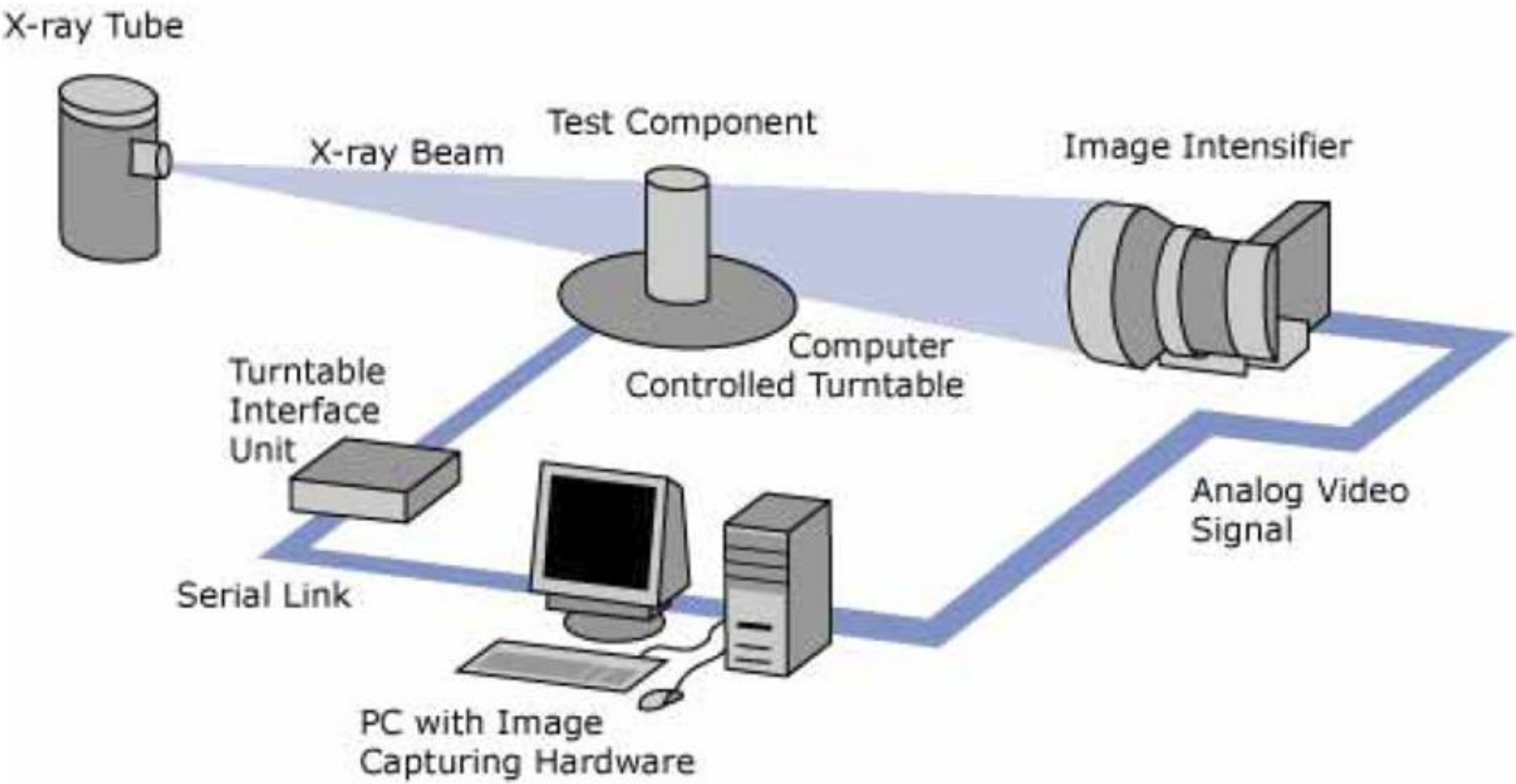


CT basic principle:

Quantification of the tendency of objects to absorb or scatter x - rays given by the optical density of the material measured in terms of attenunation coefficient

Computed Tomography, Kalender, Verlag, 2000.

CT: Imaging setup



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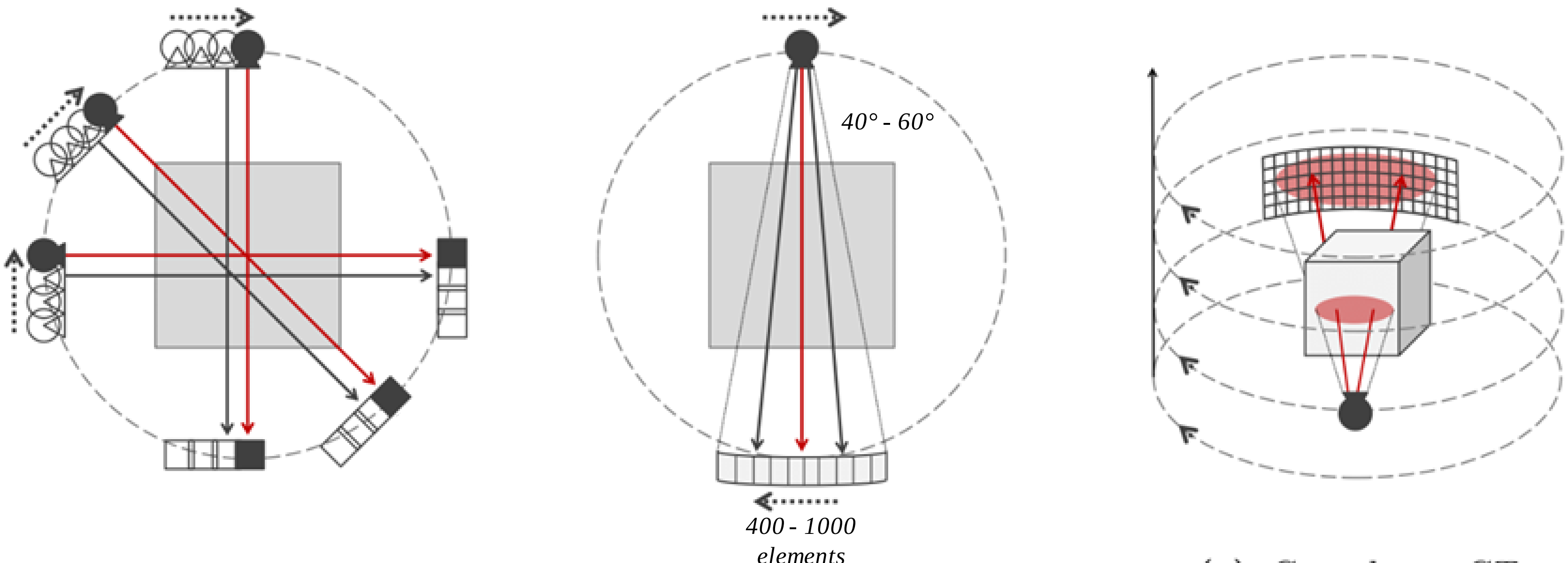
Acquisition geometry



(a) *Parallel-beam CT* **(b)** *Fan-beam CT* **(c)** *Cone-beam CT*

Figure 1.1 *Scanning geometries. In (a) a pencil X-ray source and a single detector are translated simultaneously and then rotated to take measrements through 180°. In (b) a fan of X-rays is detected by a 1D array of detectors. The apparatus rotates in a circle. In (c) a cone of X-rays is detected by a 2D array of detectors. The circular movement is supplemented by the translation in the axial direction.*

(d) *Helical (spiral) CT - cone beam geometry moving around the object in a spiral manner*

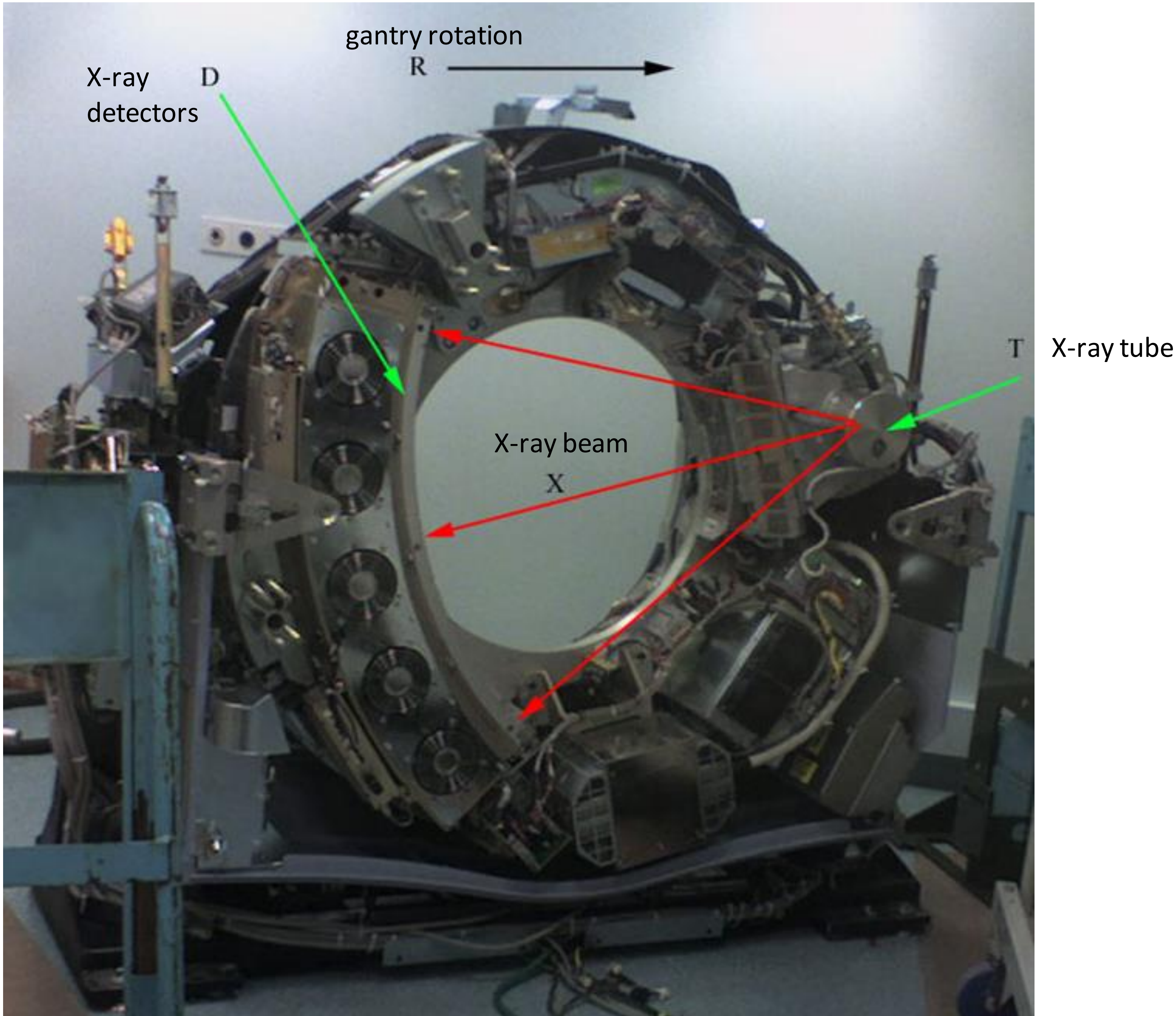

gantry rotation
R
X-ray detectors
D
X-ray tube
T
X-ray beam
X

Image reconstruction: basic concept

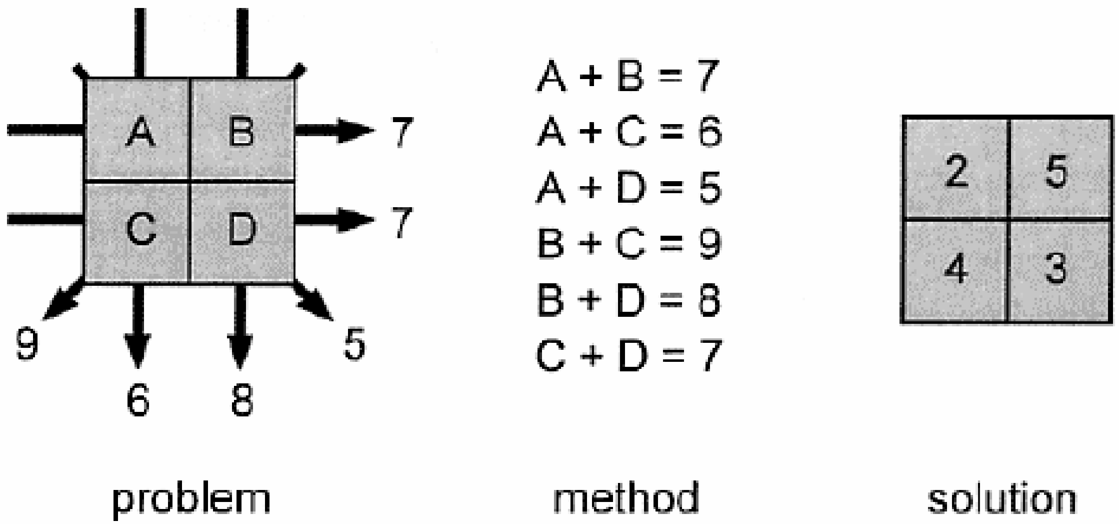
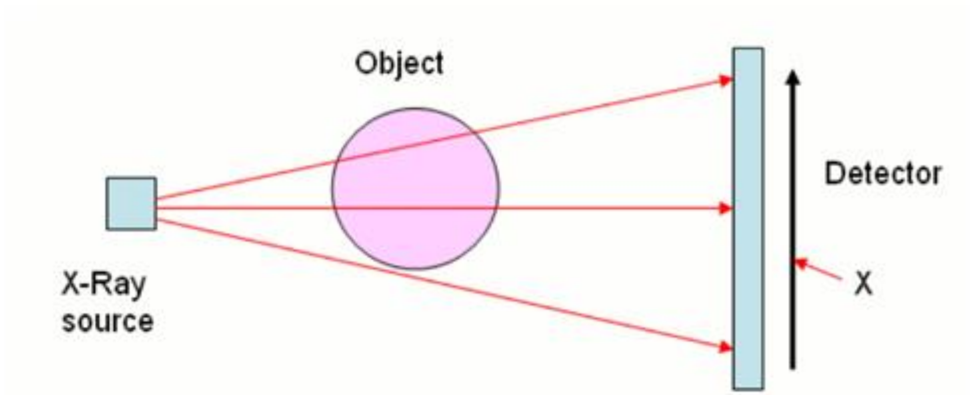


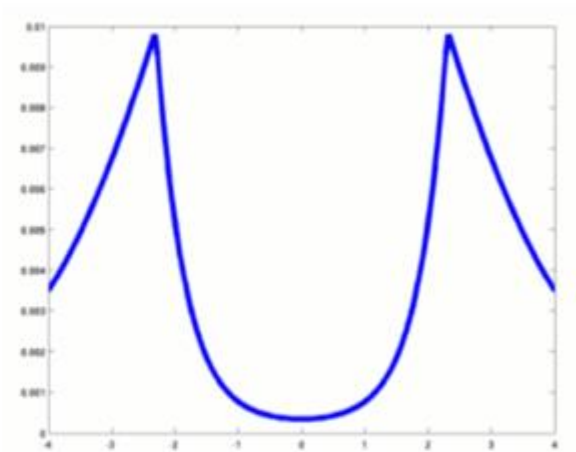
FIGURE 13-27. The mathematical problem posed by computed tomographic (CT) reconstruction is to calculate image data (the pixel values—A, B, C, and D) from the projection values *(arrows)*. For the simple image of four pixels shown here, algebra can be used to solve for the pixel values. With the six equations shown, using substitution of equations, the solution can be determined as illustrated. For the larger images of clinical CT, algebraic solutions become unfeasible, and filtered backprojection methods are used.

The Essential Physics of Medical Imaging, Bushberg

Image acquisition: basic model



The intensity of the X-ray where it hits the detector depends on the width of object and the length of the path travelled both through the object and the air.



This graph shows the intensity of the rays as they hit the detector. Rays that travel through the full width of the object have lowest intensity, as we can see from the dip in the middle of the graph. Rays that just miss the body have the highest intensity, because of all rays that are not absorbed they travel the shortest distance. This is reflected by the two spikes of the graph. Towards the edges the graph falls off, reflecting the fact that the corresponding rays have travelled a comparatively long distance.

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Budd, 2008

Image acquisition: basic maths

X-ray - moves along straight line

- at distance *s* - **intensity *I(s)***
- X-ray travels *δs* - intensity reduced by *δI*
  - reduction depends on intensity and **optical density *u(s)*** of the material
  - for small *δs*:

$$\delta I / I(s) = - u(s)\, \delta s$$

Combining all of the contributions to the reduction in the intensity of an X-ray travelling along line *L* given by all of the parts of the body that it travels through - attenuation (reduction in intensity) given by:

$$I_{finish} = I_{start} e^{-R},$$

where

$$R = \int_L u(s) ds.$$

← line integral

$$Rf(L) = \int_L f(\mathbf{x})\, |d\mathbf{x}|$$

Radon transform of function *f (x,y)*

The Radon Transform

Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten.

Von

Johann Radon.

Integriert man eine geeigneten Regularitätsbedingungen unterworfene Funktion zweier Veränderlichen x, y — eine *Punktfunktion* $f(P)$ in der Ebene — längs einer beliebigen Geraden g, so erhält man in den Integralwerten $F(g)$ eine *Geradenfunktion*. Das in Abschnitt A vorliegender Abhandlung gelöste Problem ist die Umkehrung dieser linearen Funktionaltransformation, d. h. es werden folgende Fragen beantwortet: kann jede, geeigneten Regularitätsbedingungen genügende Geradenfunktion auf diese Weise entstanden gedacht werden? Wenn ja, ist dann f durch F eindeutig bestimmt und wie kann es ermittelt werden?

Ber. Sächs. Akad. Wiss. Leipzig, Math. Phys. Kl. 69, 262 (1917)
English translation in: Deans, S.R. (1983) The Radon transform and its applications. John Wiley & Sons, NY)

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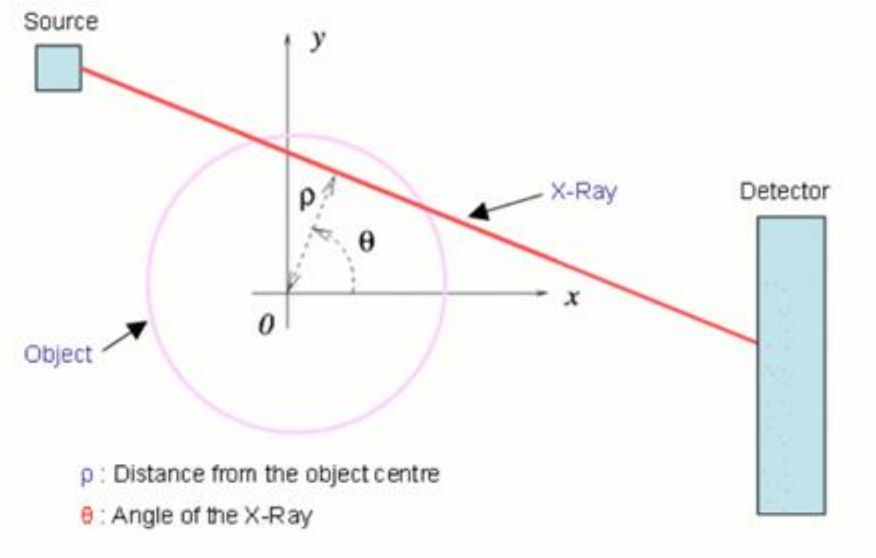
Radon transform



This X-Ray will pass through a series of points (x, y) at which the optical density is $u(x, y)$. Using the equation for a straight line these points are given by

$$(x, y) = (\rho\ \cos(\theta) - s\ \sin(\theta), \rho\ \sin(\theta) + s\ \cos(\theta)),$$

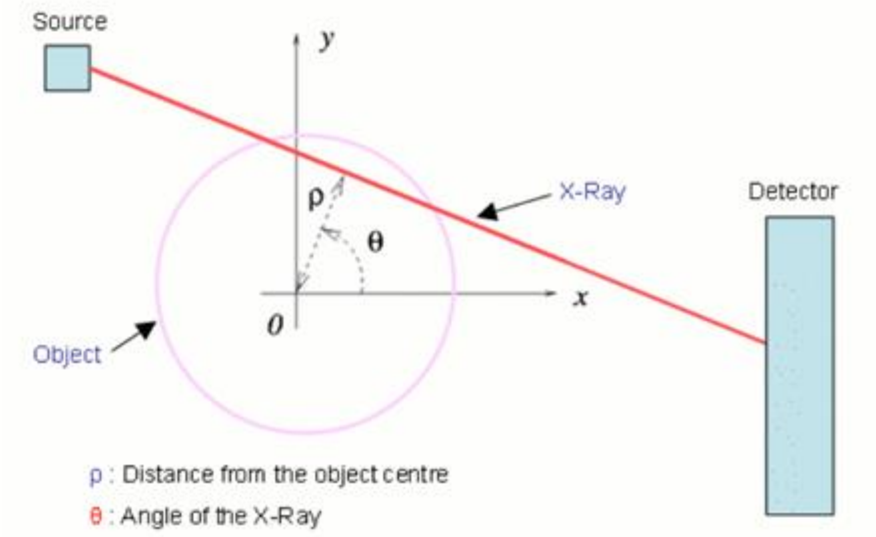
where s is the distance along the X-ray. In this case we now have

$$I_{finish} = I_{start} e^{-R(\rho,\theta)},$$

where

$$R(\rho, \theta) = \int u(\rho\ \cos(\theta) - s\ \sin(\theta), \rho\ \sin(\theta) + s\ \cos(\theta)) ds.$$

Radon transform



This X-Ray will pass through a series of points (x, y) at which the optical density is $u(x, y)$. Using the equation for a straight line these points are given by

$$(x, y) = (\rho\ \cos(\theta) - s\ \sin(\theta), \rho\ \sin(\theta) + s\ \cos(\theta)),$$

s = arc length

where s is the distance along the X-ray. In this case we now have

$$I_{finish} = I_{start} e^{-R(\rho,\theta)},$$

where

$$R(\rho, \theta) = \int u(\rho\ \cos(\theta) - s\ \sin(\theta), \rho\ \sin(\theta) + s\ \cos(\theta))ds.$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y)\ \delta(\rho - x\cos\theta - y\sin\theta)\ dx\ dy$$

Radon transform of function *u(x,y)*

δ(x,y) = Dirac Delta function

The Radon Transform

We will use the coordinate system defined in Fig. 3.1 to describe line integrals and projections. In this example the object is represented by a two-dimensional function $f(x, y)$ and each line integral by the (θ, t) parameters.

The equation of line AB in Fig. 3.1 is

$$x \cos \theta + y \sin \theta = t \qquad (1)$$

and we will use this relationship to define line integral $P_\theta(t)$ as

$$P_\theta(t) = \int_{(\theta,t)\ \text{line}} f(x, y)\, ds. \qquad (2)$$

Using a delta function, this can be rewritten as

$$P_\theta(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x \cos \theta + y \sin \theta - t)\, dx\, dy. \qquad (3)$$

The function $P_\theta(t)$ is known as the Radon transform of the function $f(x, y)$.

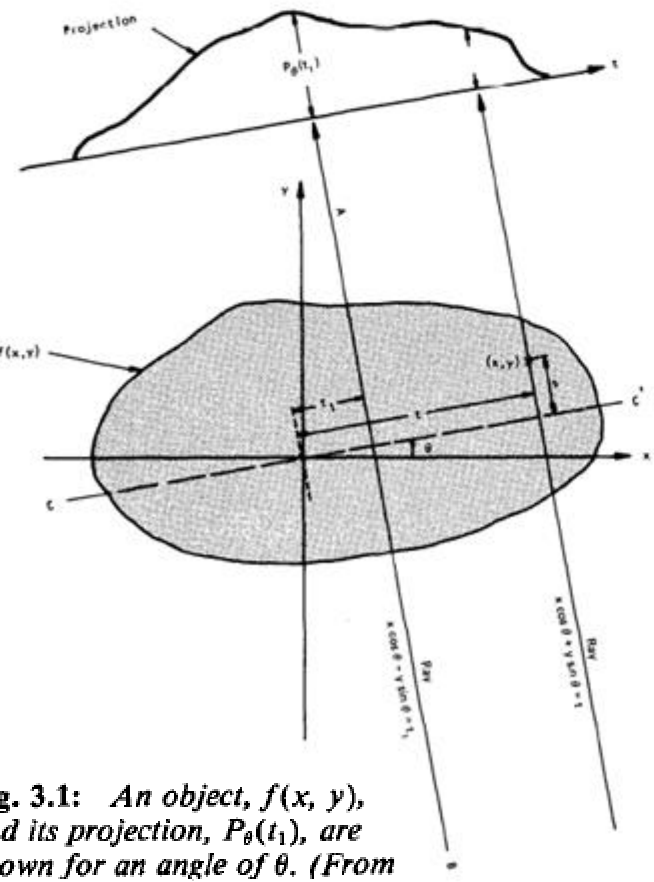


Fig. 3.1: *An object, $f(x, y)$, and its projection, $P_\theta(t_1)$, are shown for an angle of θ. (From [Kak79].)*

Basis images?

Chapter 3, Principles of Computerized Tomographic Imaging, Kak and Slaney, IEEE Press

Radon transform: properties

Let $Rg = \breve{g}(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\delta(\rho - x\cos\theta - y\sin\theta)\ dx\ dy$

1. **Linearity:**

$$g(x,y) = \sum_q \alpha_q\ g_q(x,y) \Rightarrow \breve{g}(\rho,\theta) = \sum_q \alpha_q\ \breve{g}_q(\rho,\theta)$$

1. **Shifting:**

Assume that a function $g(x,y)$ is shifted

$$\begin{aligned} h(x,y) &= g(x-x_0, y-y_0) \ \Rightarrow \\ \breve{h}(\rho,\theta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-x_0,y-y_0)\ \delta(\rho - x\cos\theta - y\sin\theta)\ dx\ dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tilde{x},\tilde{y})\ \delta((\rho - x_0\cos\theta - y_0\sin\theta) - \tilde{x}\cos\theta - \tilde{y}\sin\theta)\ d\tilde{x}\ d\tilde{y} \\ &= \breve{g}(\rho - x_0\cos\theta - y_0\sin\theta, \theta) \end{aligned}$$

Note that only the ρ-coordinate is changed.

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Toft 1996

Radon transform: properties

3. Rotation:

Here $g(x,y)$ is expressed in polar form, i.e., $g(x,y) = g(r,\phi)$. In this case rotation is fairly easy

$$\begin{aligned} h(r,\phi) &= g(r,\phi-\phi_0) \\ \check{h}(\rho,\theta) &= \int_{-\infty}^{\infty}\int_0^{\pi} g(r,\phi-\phi_0)\ \delta(\rho - r\cos\phi\cos\theta - r\sin\phi\sin\theta)\ |r|\ d\phi\ dr \\ &= \int_{-\infty}^{\infty}\int_0^{\pi} \check{g}(r,\tilde{\phi})\ \delta(\rho - r\cos(\theta-\tilde{\phi}-\phi_0))\ |r|\ d\tilde{\phi}\ dr \\ &= \check{g}(\rho,\theta-\phi_0) \end{aligned}$$

This is quite obvious. If the coordinate system (x,y) is turned ϕ_0, then the Radon transform is also turned ϕ_0.

4. Convolution

Assume the function $h(x,y)$ being a 2D convolution of $f(x,y)$ and $g(x,y)$.

$$h(x,y) = f(x,y) * *g(x,y) = \int\int f(x_1,y_1)\ g(x-x_1,y-y_1)\ dx_1\ dy_1$$

$$\begin{aligned} \check{h}(\rho,\theta) &= \int_{-\infty}^{\infty} \check{f}(\rho_1,\theta)\ \check{g}(\rho-\rho_1,\theta)\ d\rho_1 \\ &= \check{f}(\rho,\theta)\ *\ \check{g}(\rho,\theta) \end{aligned}$$

Radon transform of a 2D convolution is a 1D convolution of the Radon transformed functions with respect to ρ

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Radon Transform: Point source

Here an arbitrary position of the point source (x^*, y^*) is assumed.

$$
\begin{aligned}
g(x,y) &= \delta(x-x^*)\ \delta(y-y^*) \ \Rightarrow \\
\breve{g}(\rho,\theta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x^*)\ \delta(y-y^*)\ \delta(\rho - x\cos\theta - y\sin\theta)\ dx\ dy \\
&= \delta(\rho - x^*\cos\theta - y^*\sin\theta)
\end{aligned}
$$

Fig. 2.2 illustrates a point source and the corresponding Radon transform.

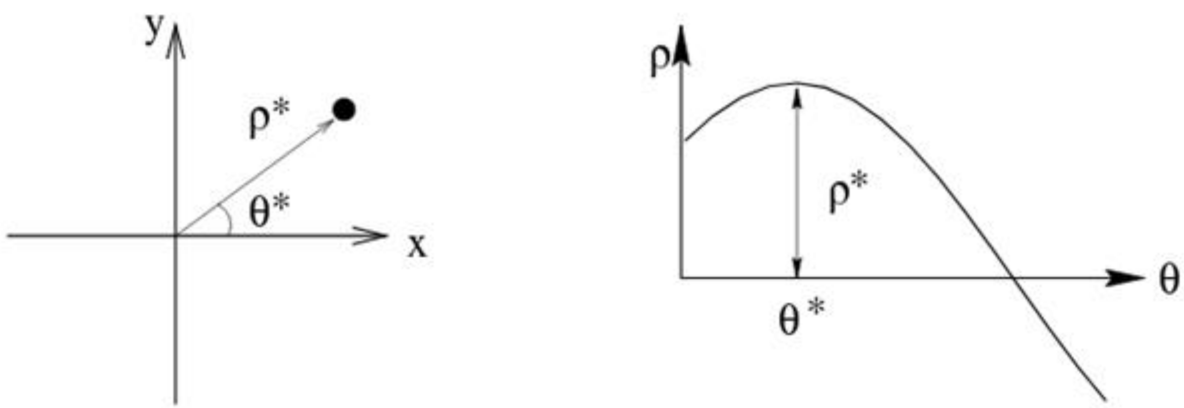
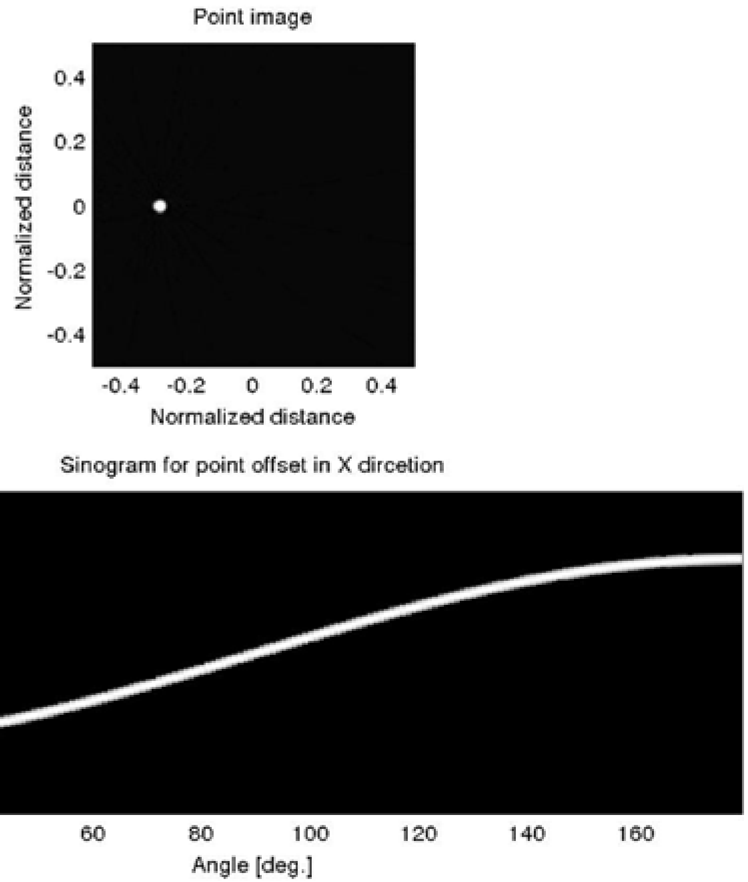
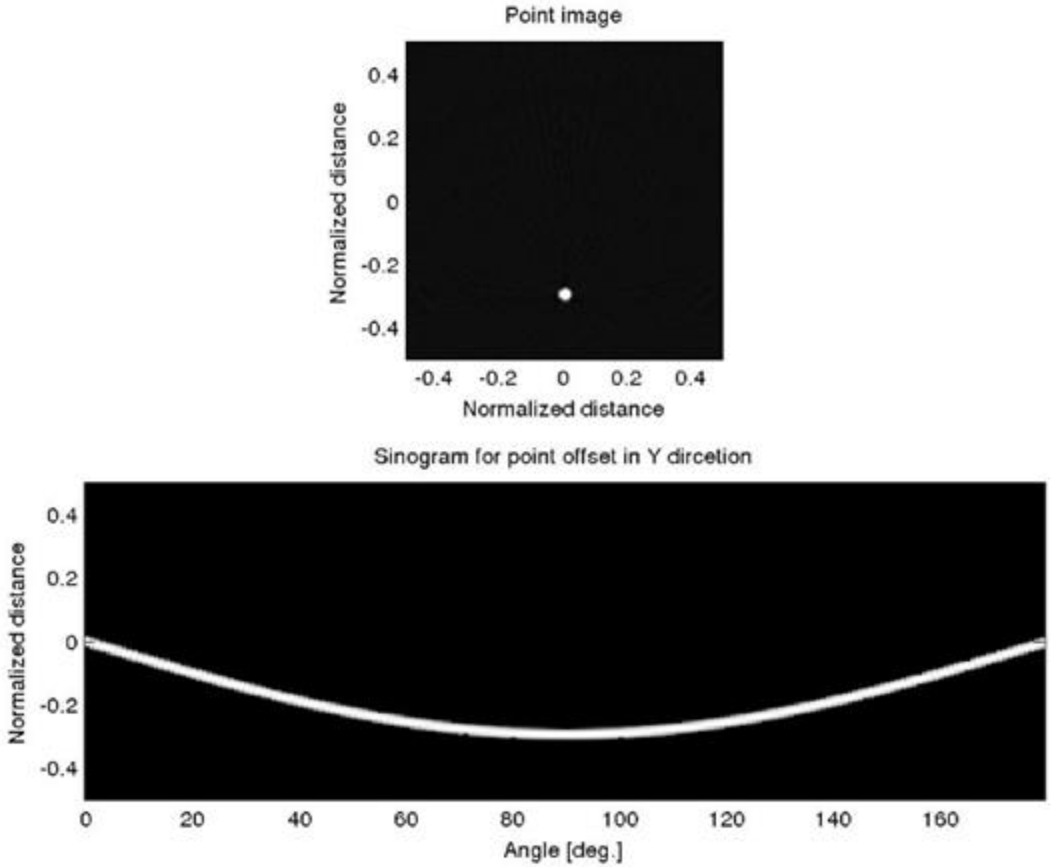


Figure 2.2 To the left is shown a point source, and to the right is shown the corresponding normal Radon transform.

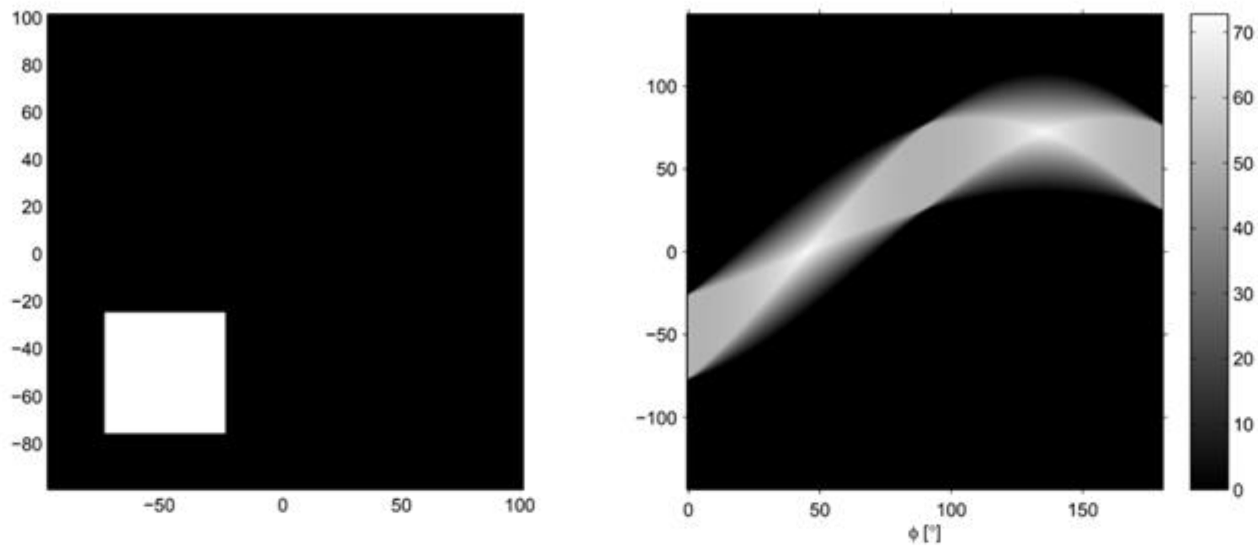
Radon Transform: Sinogram



Radon Transform: Sinogram



Radon Transform: Sinogram



f $\mathcal{R}f$

Figure 1.3 *Radon transform (sinogram) of a two-dimensional function f. (Left) Note that the origin is located at the centre of the image. (Right) High-intensity points correspond to diagonals of the square.*

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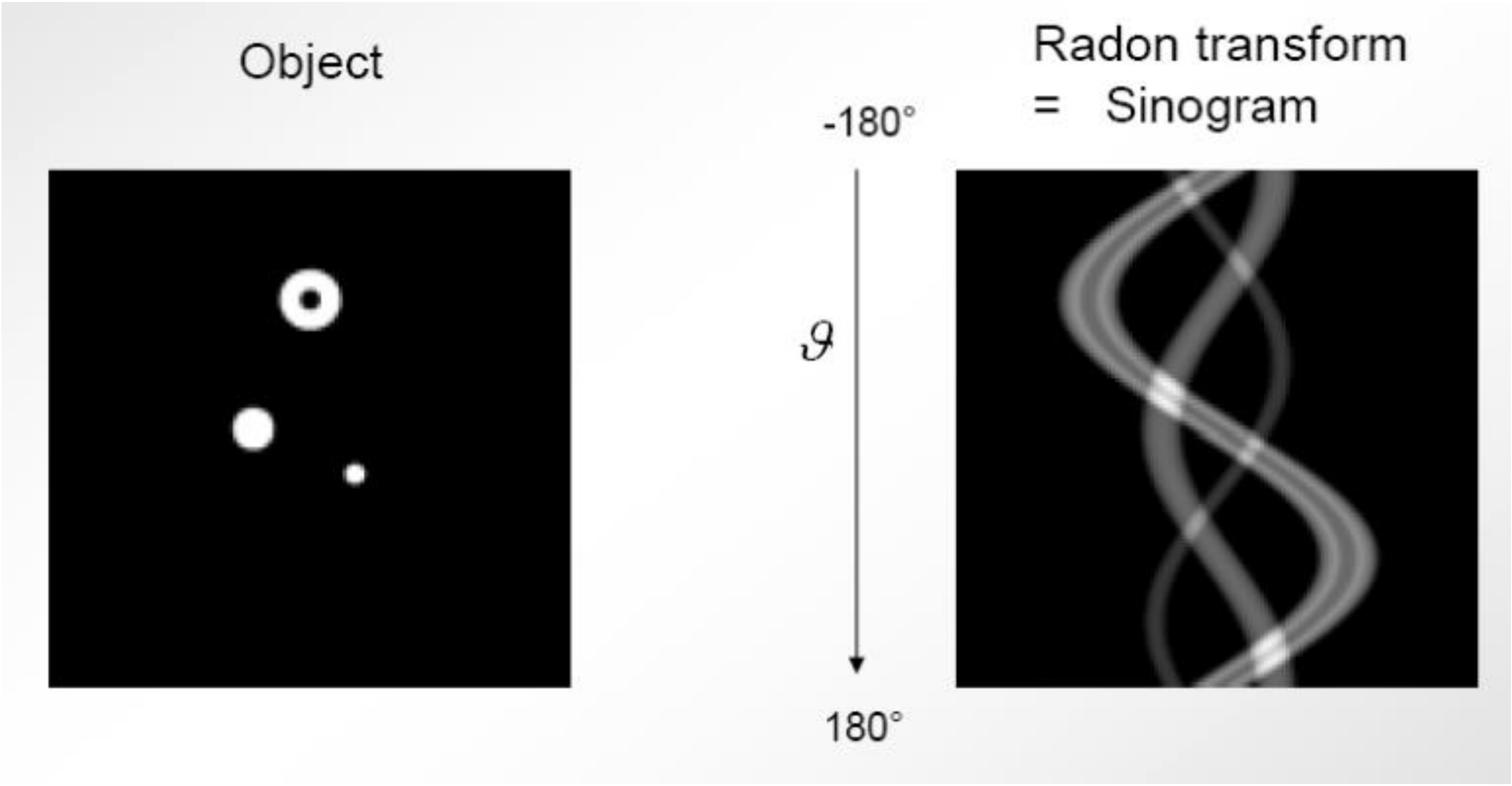
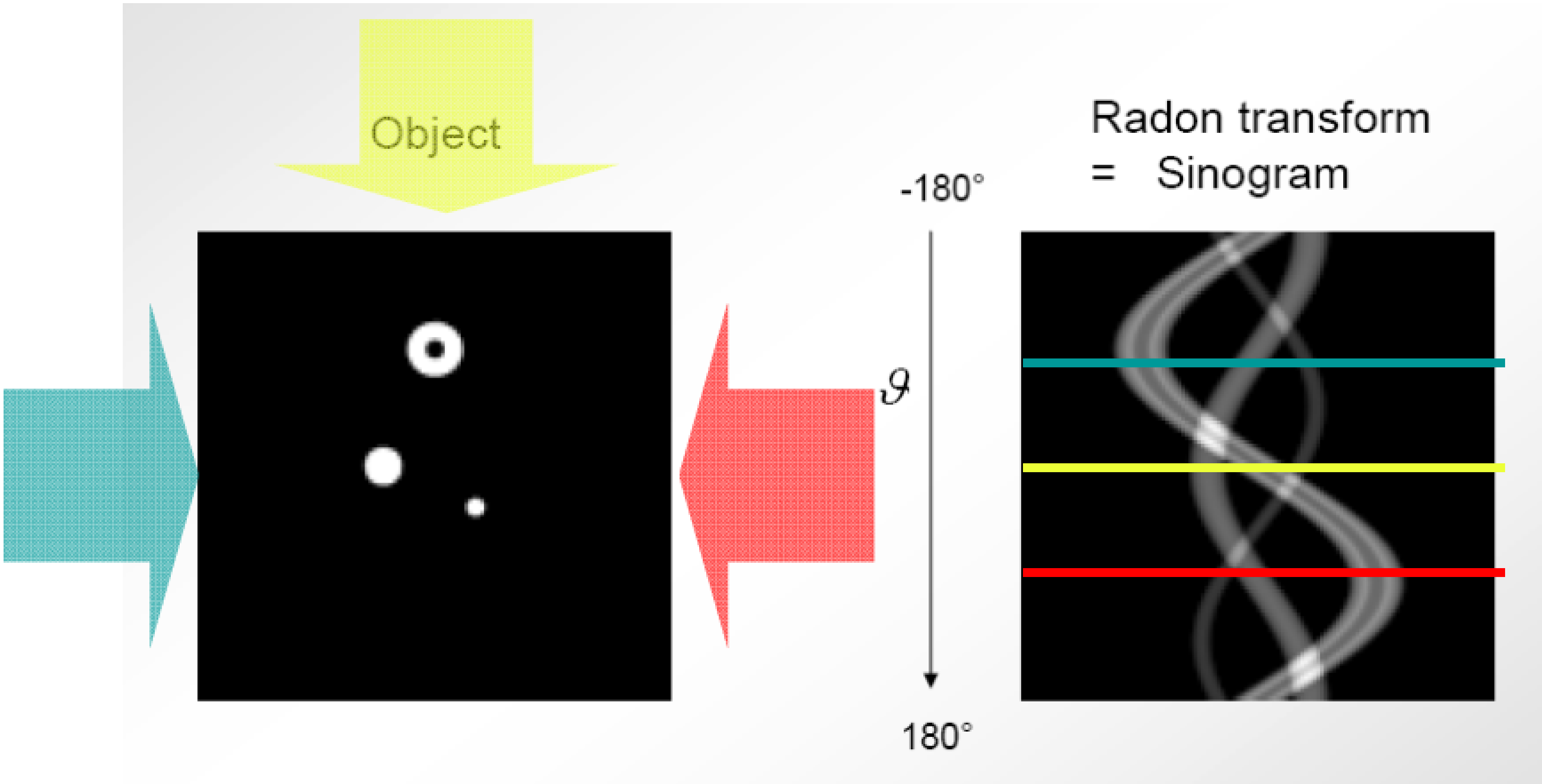
Radon Transform: Sinogram



Radon Transform: Sinogram



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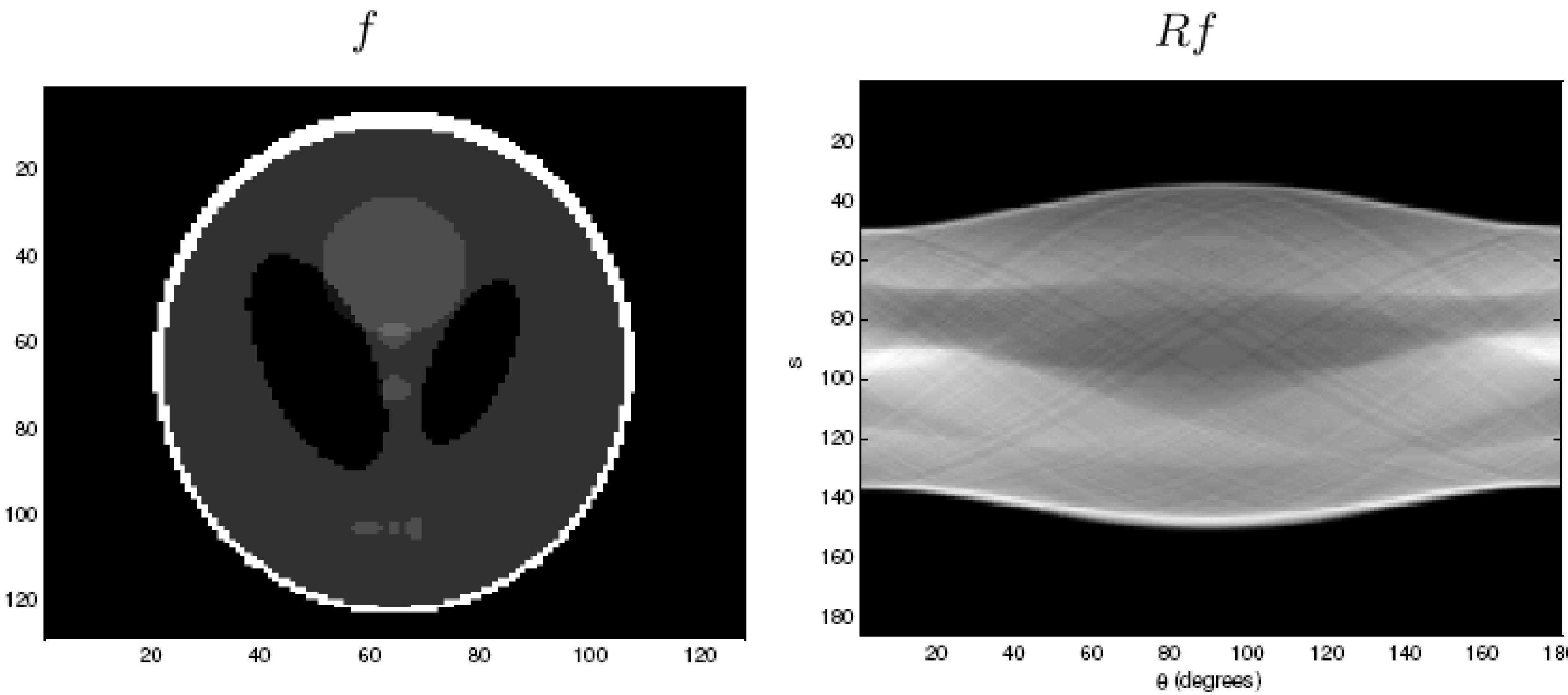
Radon Transform: Sinogram



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Question:

Can we find the function $u(x,y)$ if we know the function $R(\rho,\theta)$?

The Radon Transform

Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten.

Von

JOHANN RADON.

Integriert man eine geeigneten Regularitätsbedingungen unterworfene Funktion zweier Veränderlichen x, y — eine *Punktfunktion* $f(P)$ in der Ebene — längs einer beliebigen Geraden g, so erhält man in den Integralwerten $F(g)$ eine *Geradenfunktion*. Das in Abschnitt A vorliegender Abhandlung gelöste Problem ist die Umkehrung dieser linearen Funktionaltransformation, d. h. es werden folgende Fragen beantwortet: kann jede, geeigneten Regularitätsbedingungen genügende Geradenfunktion auf diese Weise entstanden gedacht werden? Wenn ja, ist dann f durch F eindeutig bestimmt und wie kann es ermittelt werden?

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Image reconstruction: Algebraic formulation

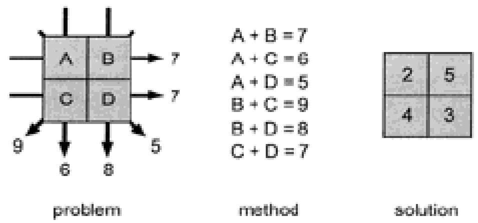

FIGURE 13-27. The mathematical problem posed by computed tomographic (CT) reconstruction is to calculate image data (the pixel values—A, B, C, and D) from the projection values (*arrows*). For the simple image of four pixels shown here, algebra can be used to solve for the pixel values. With the six equations shown, using substitution of equations, the solution can be determined as illustrated. For the larger images of clinical CT, algebraic solutions become unfeasible, and filtered backprojection methods are used.

$$\begin{pmatrix} 1\,1\,0\,0 \\ 1\,0\,1\,0 \\ 1\,0\,0\,1 \\ 0\,1\,1\,0 \\ 0\,1\,0\,1 \\ 0\,0\,1\,1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 5 \\ 9 \\ 8 \\ 7 \end{pmatrix}$$

tomography matrix
$K = \{k_{ij}\}$

Tomography system through the lense of linear algebra:

- Assumption - attenuation of material within each pixel constant and proportional to the area of the pixel illuminated by the beam

$$k_{ij} = \frac{\text{area of pixel } j \text{ illuminated by ray } i}{\text{total area of pixel } j},$$

$i = 1, \dots, l,\ j = 1, \dots, nm.$
Hence, the algebraic model reads

$$\boxed{K\boldsymbol{f} = \boldsymbol{g},}$$

f = BW plane/volumetric image to be retrieved (reshaped into a vector)
g = attenuation measurements from the CT system

The Essential Physics of Medical Imaging, Bushberg

Image reconstruction: Algebraic formulation

Overdetermined non-square matrix K

$$K f = g,$$

Transform into a **system of normal equations**

$$K^T K f = K^T g$$

Ill-posed problem:
Hadamard - solution does not exist, is not unique or not continuously dependent on data

Methods:
Large systems need to be solved iteratively:

- Algebraic Reconstruction technique (ART), Simultaneous Iterative Reconstruction Techniques (SIRT), Simultaneous Algebraic Reconstruction Technique(SART)
- direct methods - Tikhonov regularisation, SVD etc.

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Image reconstruction: Backprojection

Single linear projection

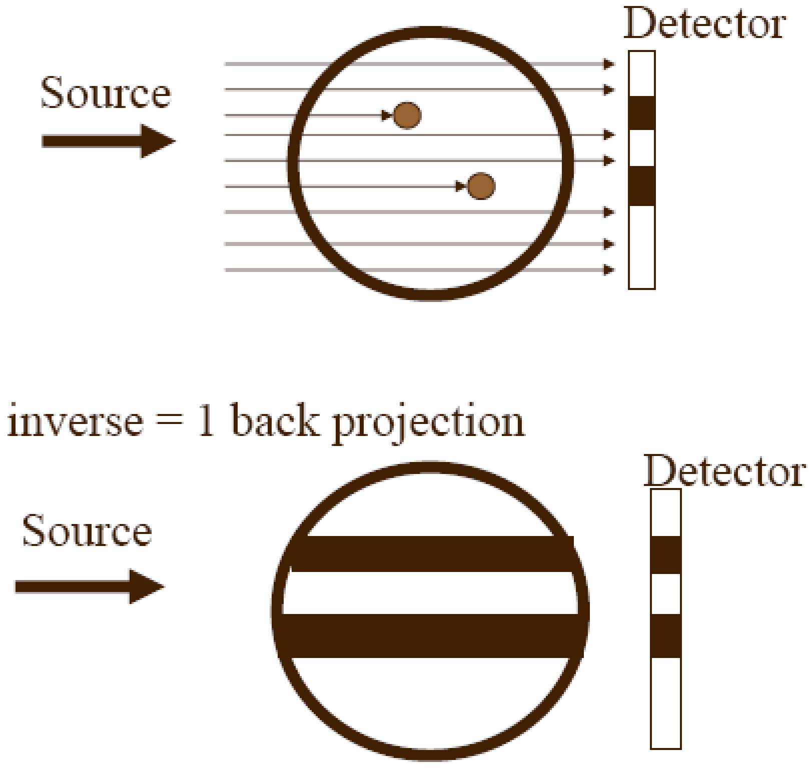


Image reconstruction: Backprojection

Two linear projections

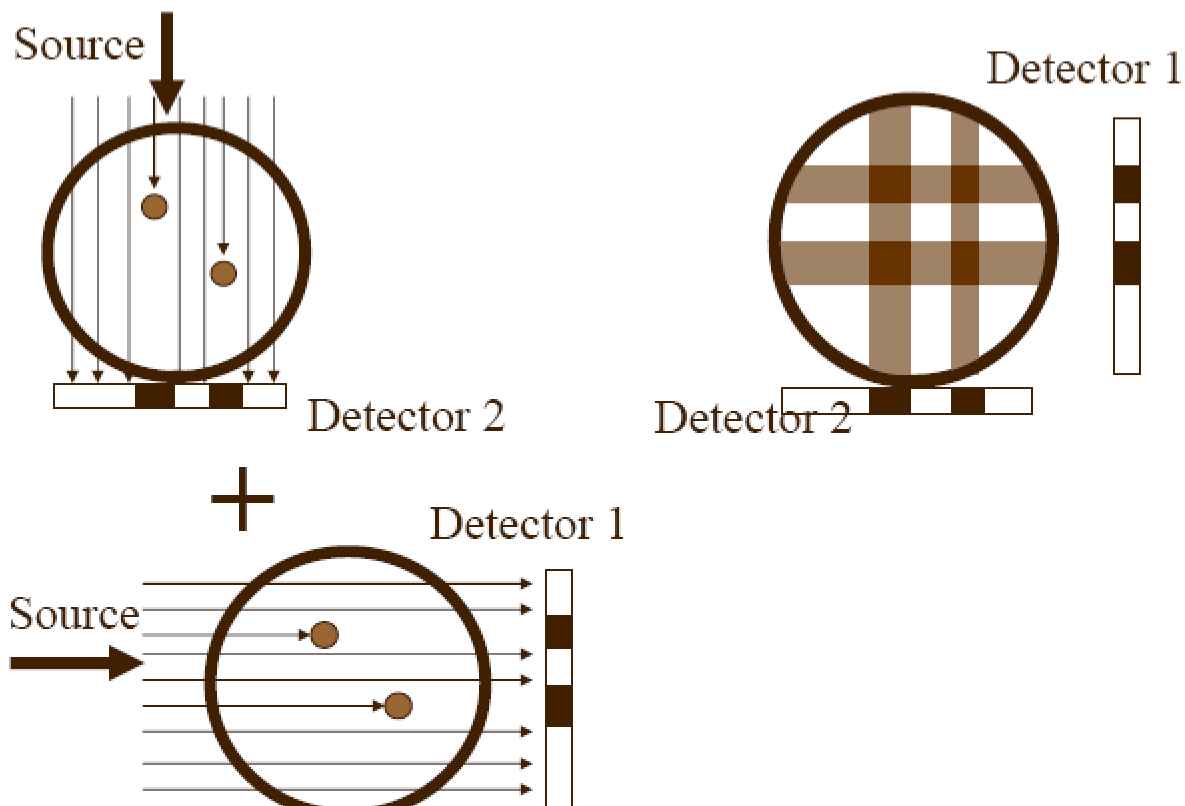
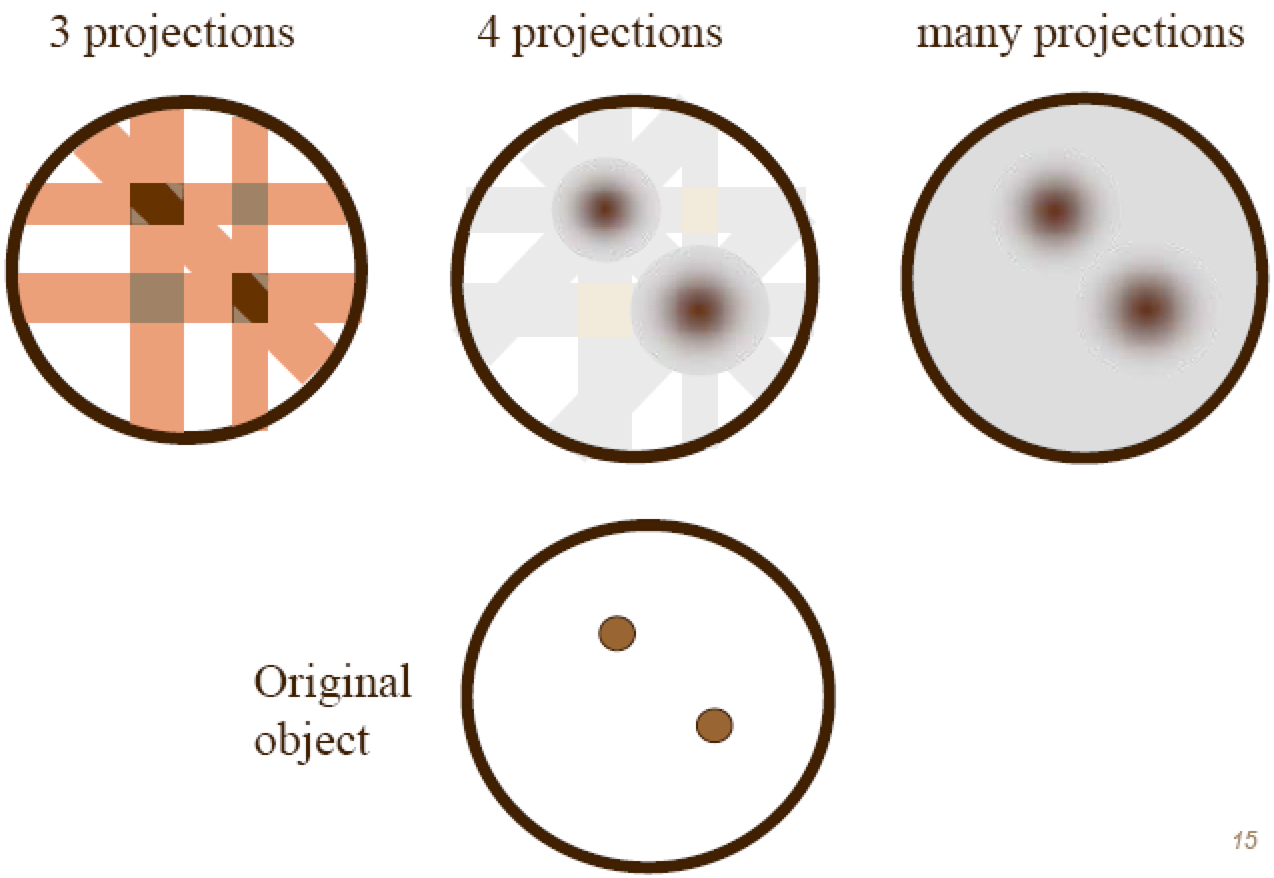
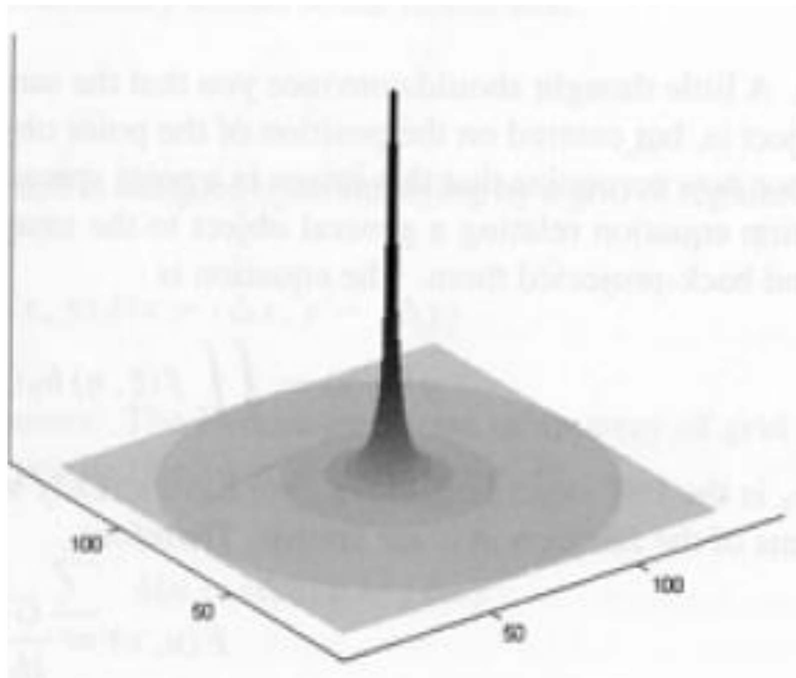


Image reconstruction: Backprojection

Multiple linear projections



Backprojection of point



Fourier / Central Slice Theorem

- Facilitates inversion of Radon transform

$$\mathcal{G}(q, 0) = \mathcal{F}(q \cos 0, q \sin 0)$$

G = 1D Fourier transform of the attenuation measurements *g = Rf* [keep θ fixed]

F = 2D Fourier transform of the object slice *f(x,y)* evaluated at a particular point

The Fourier Slice Theorem

We start by defining the two-dimensional Fourier transform of the object function as

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)}\, dx\, dy. \tag{7}$$

Likewise define a projection at an angle θ, $P_\theta(t)$, and its Fourier transform by

$$S_\theta(w) = \int_{-\infty}^{\infty} P_\theta(t) e^{-j2\pi wt}\, dt. \tag{8}$$

The simplest example of the Fourier Slice Theorem is given for a projection at $\theta = 0$. First, consider the Fourier transform of the object along the line in the frequency domain given by $v = 0$. The Fourier transform integral now simplifies to

$$F(u, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi ux}\, dx\, dy \tag{9}$$

but because the phase factor is no longer dependent on y we can split the integral into two parts,

$$F(u, 0) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x, y)\, dy \right] e^{-j2\pi ux}\, dx. \tag{10}$$

From the definition of a parallel projection, the reader will recognize the term in brackets as the equation for a projection along lines of constant x or

$$P_{\theta=0}(x) = \int_{-\infty}^{\infty} f(x, y)\, dy. \tag{11}$$

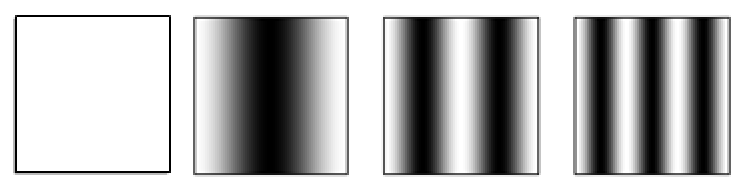
Substituting this in (10) we find

$$F(u, 0) = \int_{-\infty}^{\infty} P_{\theta=0}(x) e^{-j2\pi ux}\, dx. \tag{12}$$

The right-hand side of this equation represents the one-dimensional Fourier transform of the projection $P_{\theta=0}$; thus we have the following relationship between the vertical projection and the 2-D transform of the object function:

$$F(u, 0) = S_{\theta=0}(u). \tag{13}$$

This is the simplest form of the Fourier Slice Theorem.



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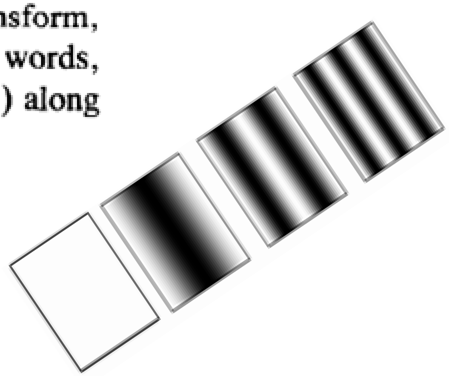
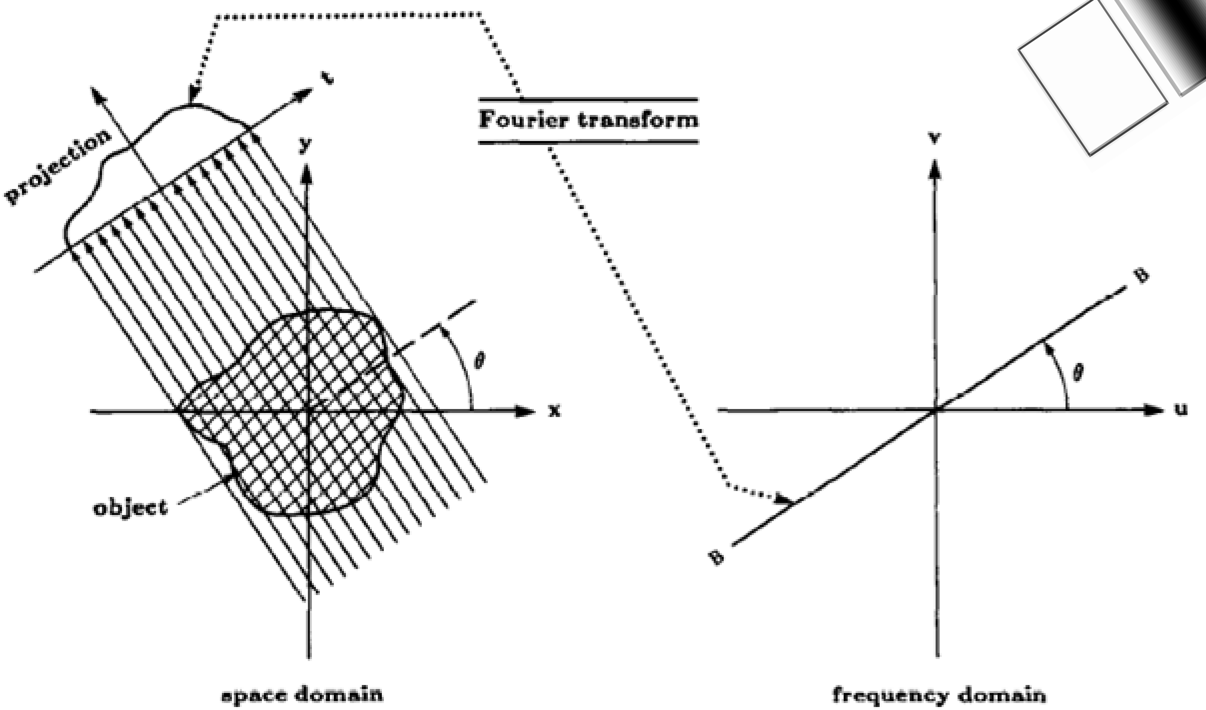
Chapter 3, Principles of Computerized Tomographic Imaging, Kak and Slaney, IEEE Press

Fourier Slice Theorem

Fig. 3.6: *The Fourier Slice Theorem relates the Fourier transform of a projection to the Fourier transform of the object along a radial line. (From [Pan83].)*

The Fourier transform of a parallel projection of an image $f(x, y)$ taken at angle θ gives a slice of the two-dimensional transform, $F(u, v)$, subtending an angle θ with the u-axis. In other words, the Fourier transform of $P_\theta(t)$ gives the values of $F(u, v)$ along line BB in Fig. 3.6.





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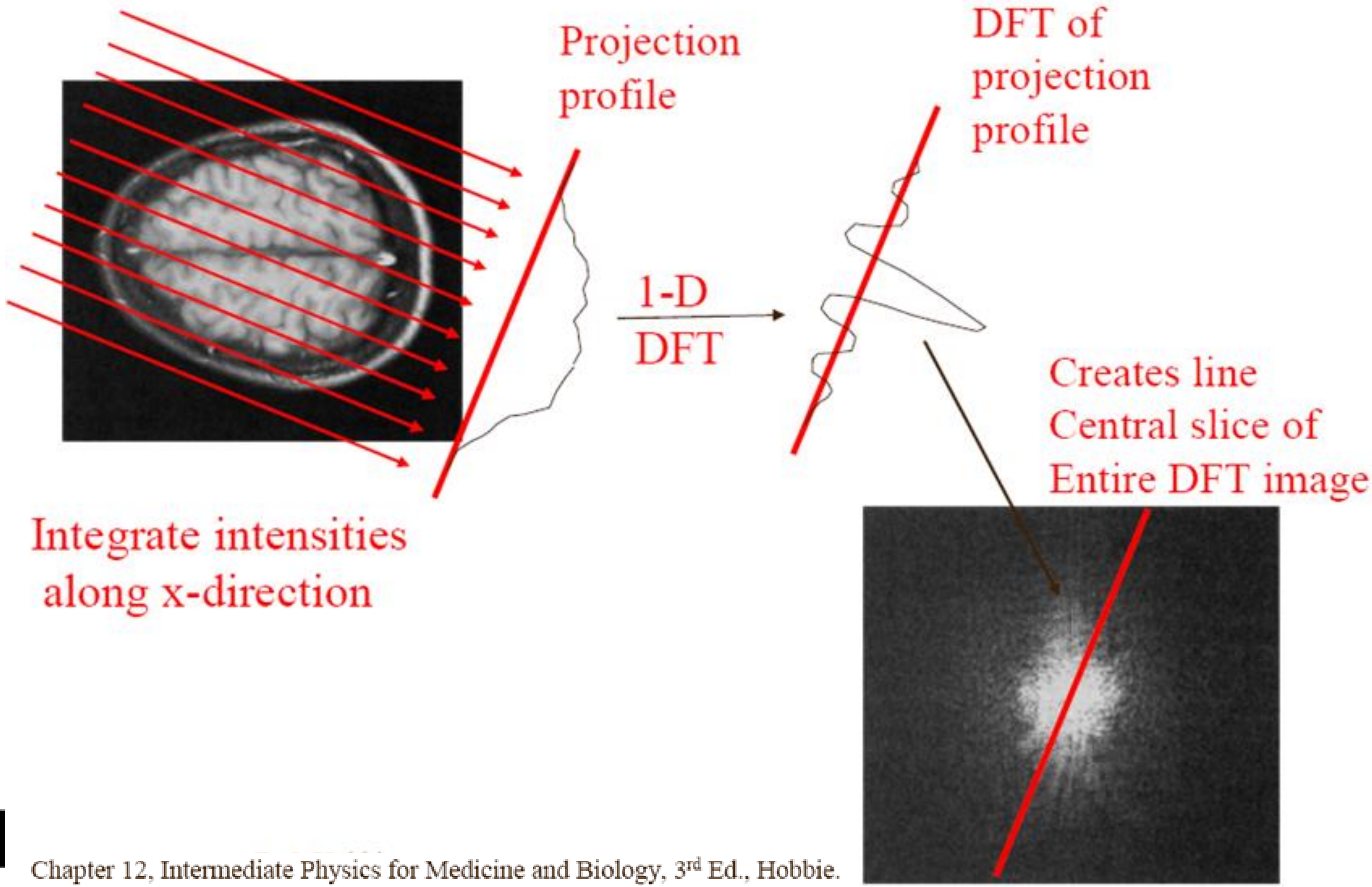
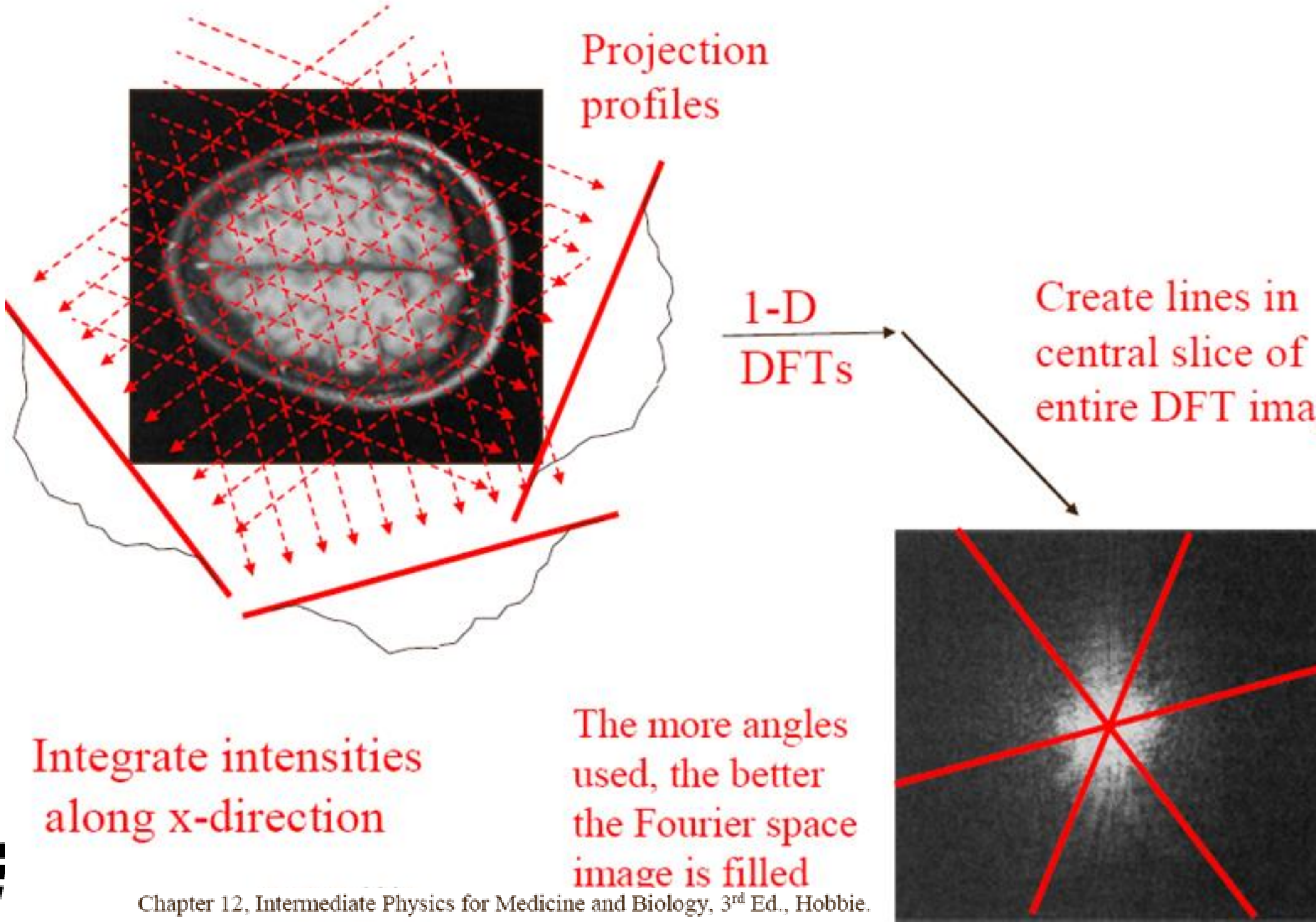
Chapter 3, Principles of Computerized Tomographic Imaging, Kak and Slaney, IEEE Press

Fourier slice theorem: What does the DFT image represent?



Chapter 12, Intermediate Physics for Medicine and Biology, 3rd Ed., Hobbie.

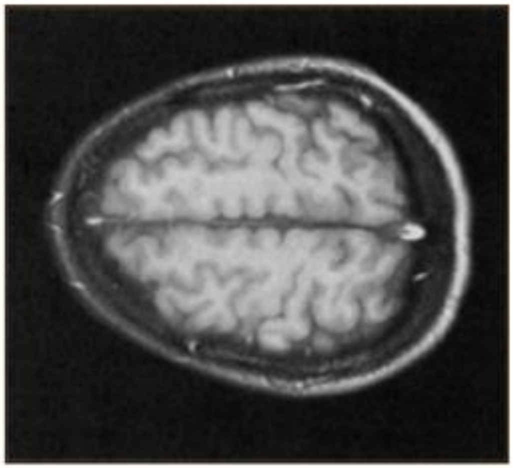
Fourier slice theorem: What does the DFT image represent?



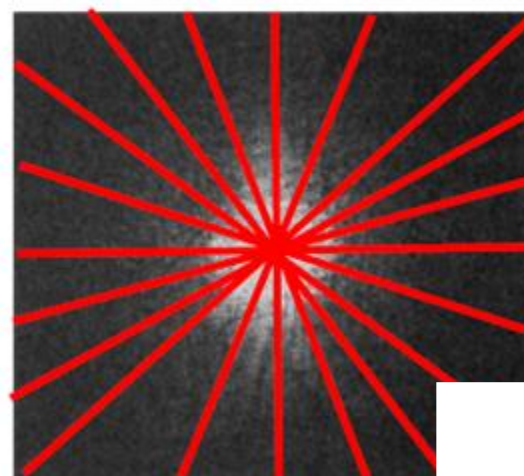
Chapter 12, Intermediate Physics for Medicine and Biology, 3rd Ed., Hobbie.

Fourier slice theorem: What does the DFT image represent?

DFT image represents integration of original projections DFT transformed and summed together.



1-D DFTs at each projection

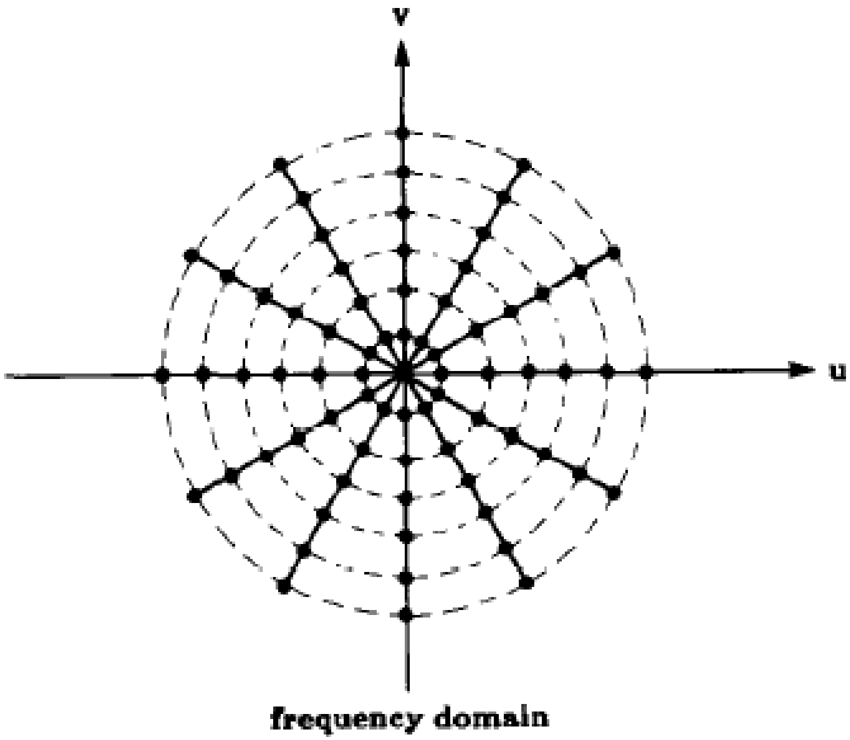


This is the fast way to create the DFT image from projection data. The more projections taken, the more complete the sampling.

Chapter 12, Intermediate Physics for Medicine and Biology, 3rd Ed., Hobbie.

Fourier slice theorem

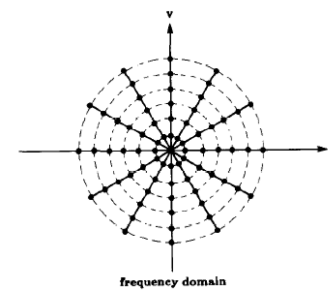
Fig. 3.7: *Collecting projections of the object at a number of angles gives estimates of the Fourier transform of the object along radial lines. Since an FFT algorithm is used for transforming the data, the dots represent the actual location of estimates of the object's Fourier transform. (From [Pan83].)*

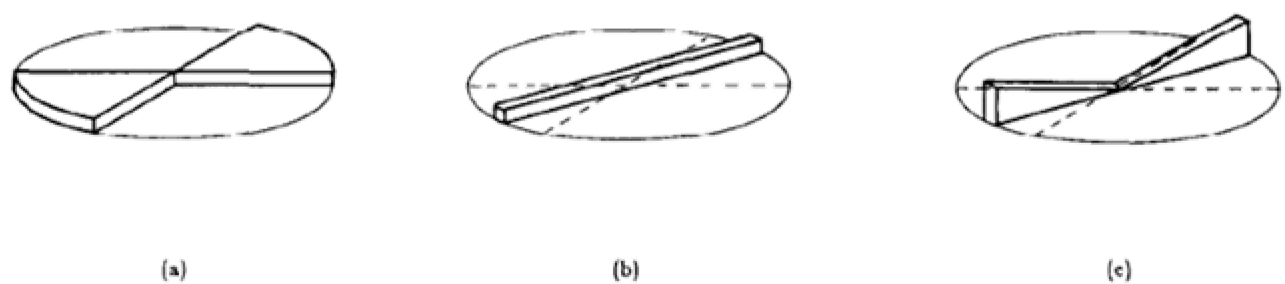


Chapter 3, Principles of Computerized Tomographic Imaging, Kak and Slaney, IEEE Press

Tomographic reconstruction: concept

Fig. 3.8: *This figure shows the frequency domain data available from one projection. (a) is the ideal situation. A reconstruction could be formed by simply summing the reconstruction from each angle until the entire frequency domain is filled. What is actually measured is shown in (b). As predicted by the Fourier Slice Theorem, a projection gives information about the Fourier transform of the object along a single line. The filtered backprojection algorithm takes the data in (b) and applies a weighting in the frequency domain so that the data in (c) are an approximation to those in (a).*





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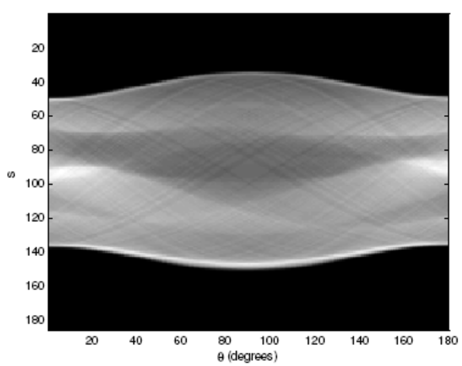
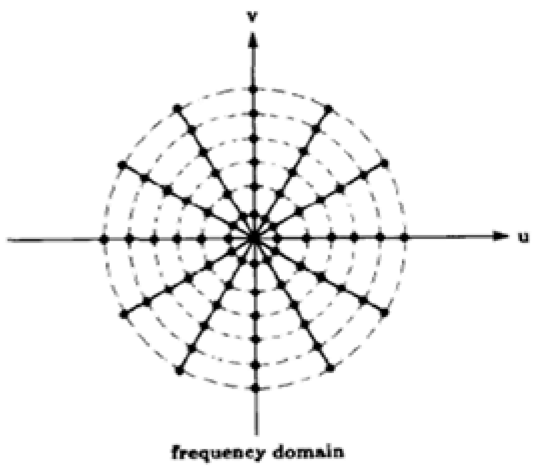
Filtered backprojection algorithm

Sum for each of the K angles, θ, between 0 and 180°
  Measure the projection, $P_\theta(t)$
  Fourier transform it to find $S_\theta(w)$
  Multiply it by the weighting function $2\pi|w|/K$
  Sum over the image plane the inverse Fourier transforms of the filtered projections (the backprojection process).





$$\Sigma$$

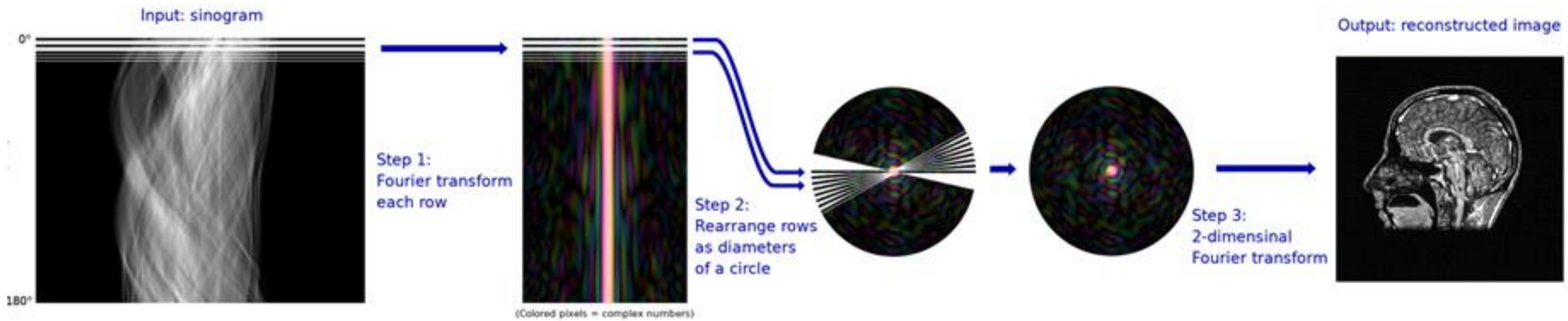
Fourier Slice Theorem: Backprojection

Perform for all projections - over all projection angles θ:

- Measure projection (attenuation) data
- 1D FT of projection data
- Make 2D inverse FT and sum with previous image(i.e. backpropagate)

In practice - issues:

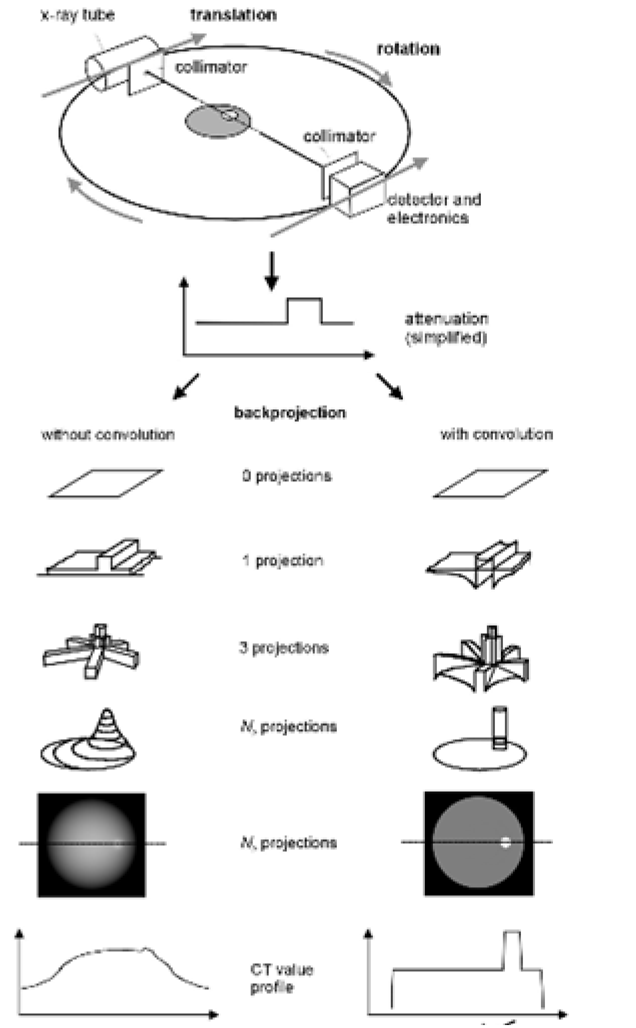
- requires many precise attenuation measurements
- sensitive to noise
- unstable & hard to implement accurately
- **blurring in the final image**



Fourier Slice Theorem: Filtered Backprojection

Perform for all projections - over all projection angles θ:

- Measure projection (attenuation) data
- 1D FT of projection data
- Apply high-pass filter in Fourier domain
- Make 2D inverse FT and sum with previous image (i.e. backproject)



Center for Fast Ultrasound Imaging, Department of Electrical Engineering Technical University of Denmark

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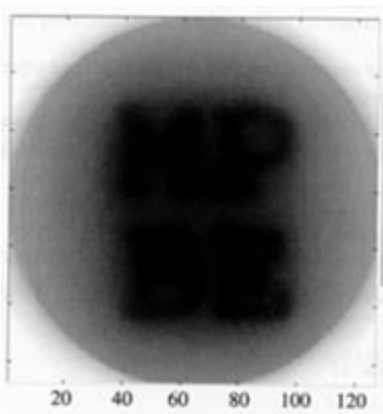
Tomographic reconstruction

backprojection



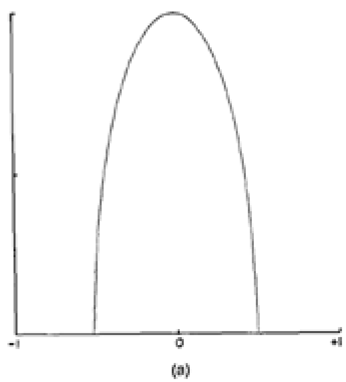
filtered backprojection

Figure 11.15. *The image reconstructed from the sinogram of figure 11.12(b). (a) The image reconstructed from the profiles collected every 30°; (b) using profiles spaced at 10°; (c) profiles spaced at 2°. The importance of taking enough profiles is apparent from these figures.*

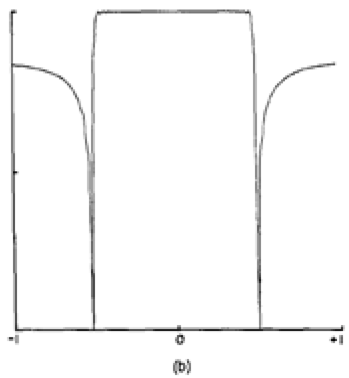
Medical Physics and Biomedical Engineering, Brown et al., IoP Publishing

Tomographic reconstruction

Filtering can be implemented as a physical filter on the CT scanner itself - low energy X-rays removed:

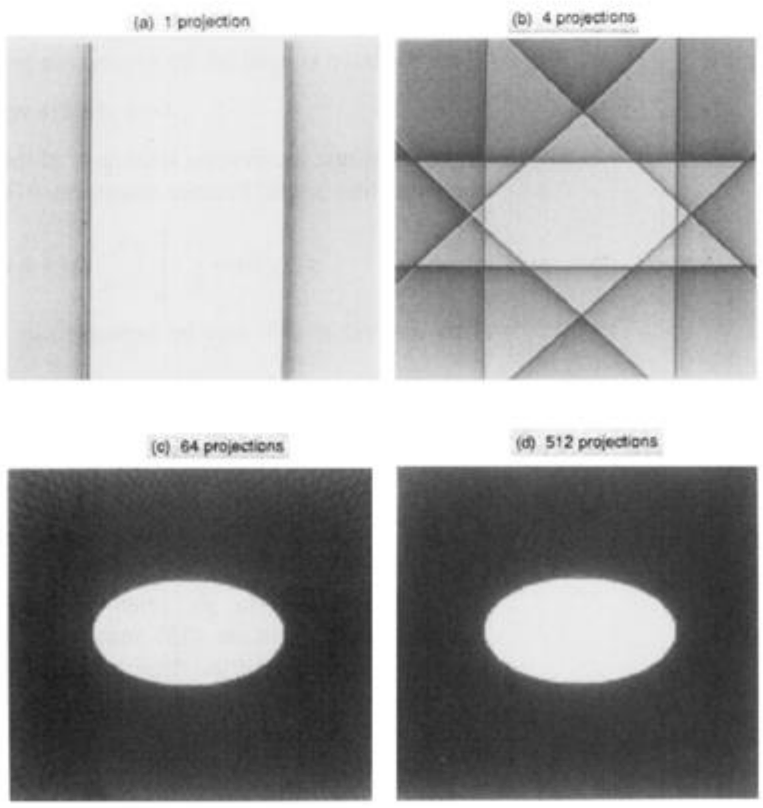


original attenuation profile

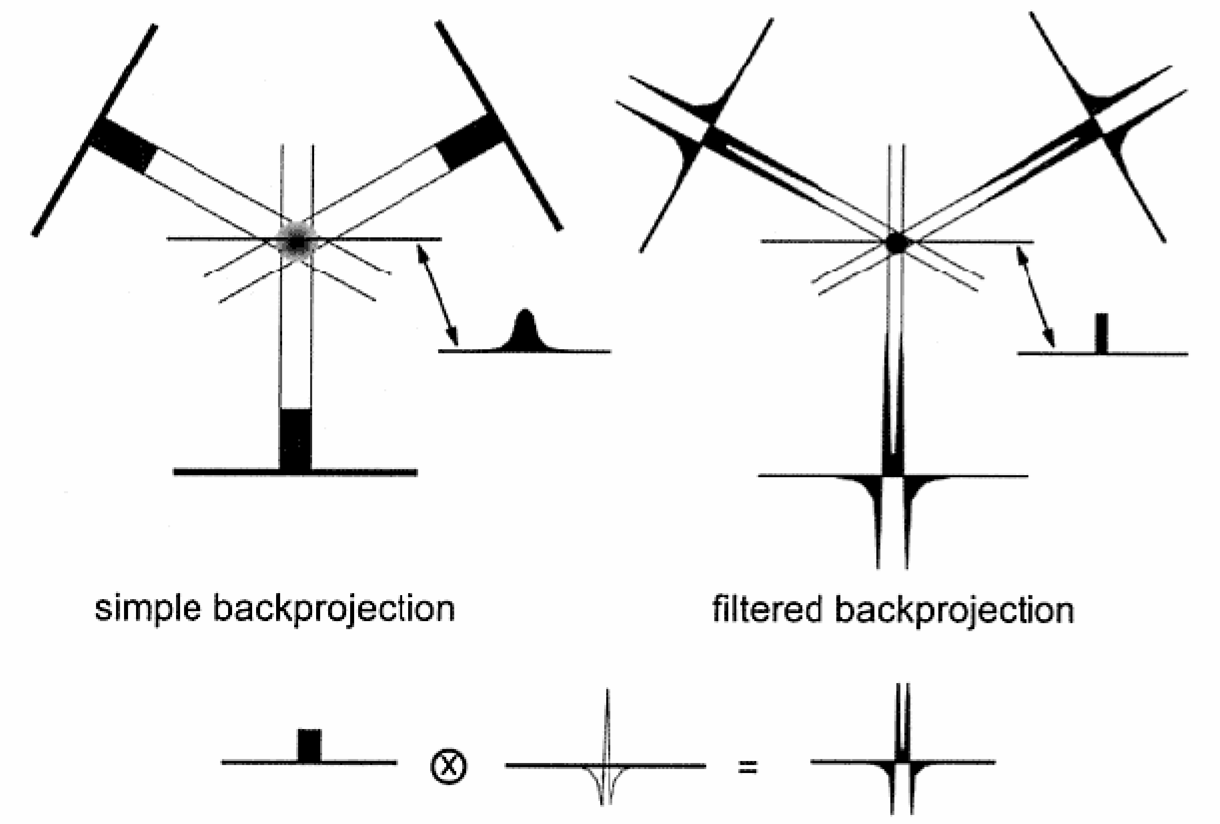


filtered attenuation profile

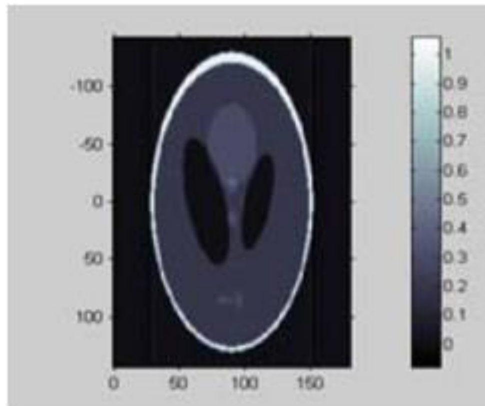
Backpropagation results:

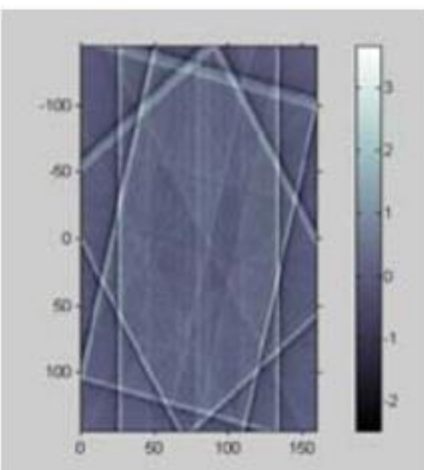


Tomographic reconstruction

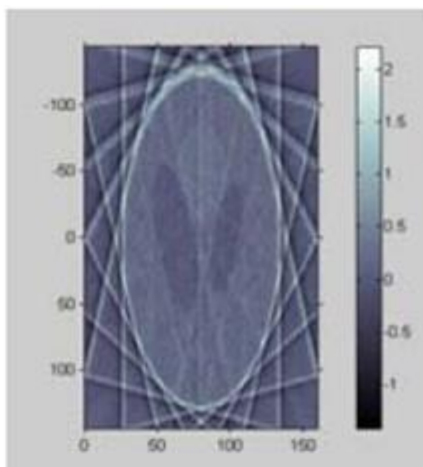


Tomographic reconstruction

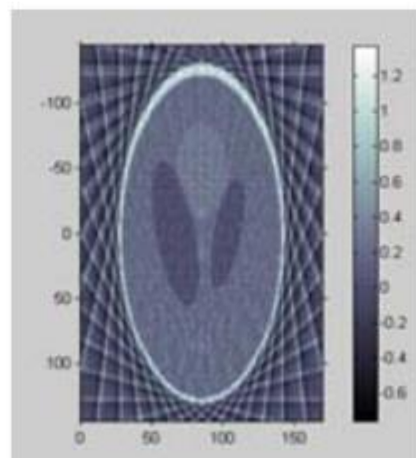




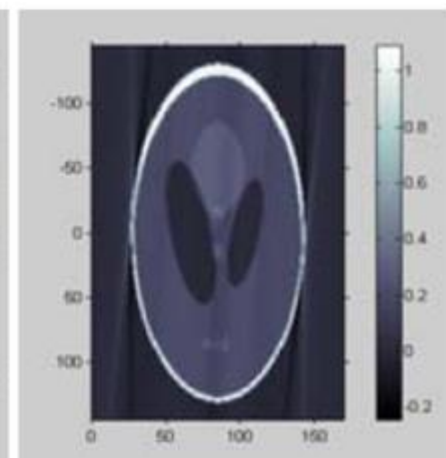
theta = 0:40:170



theta = 0:20:170

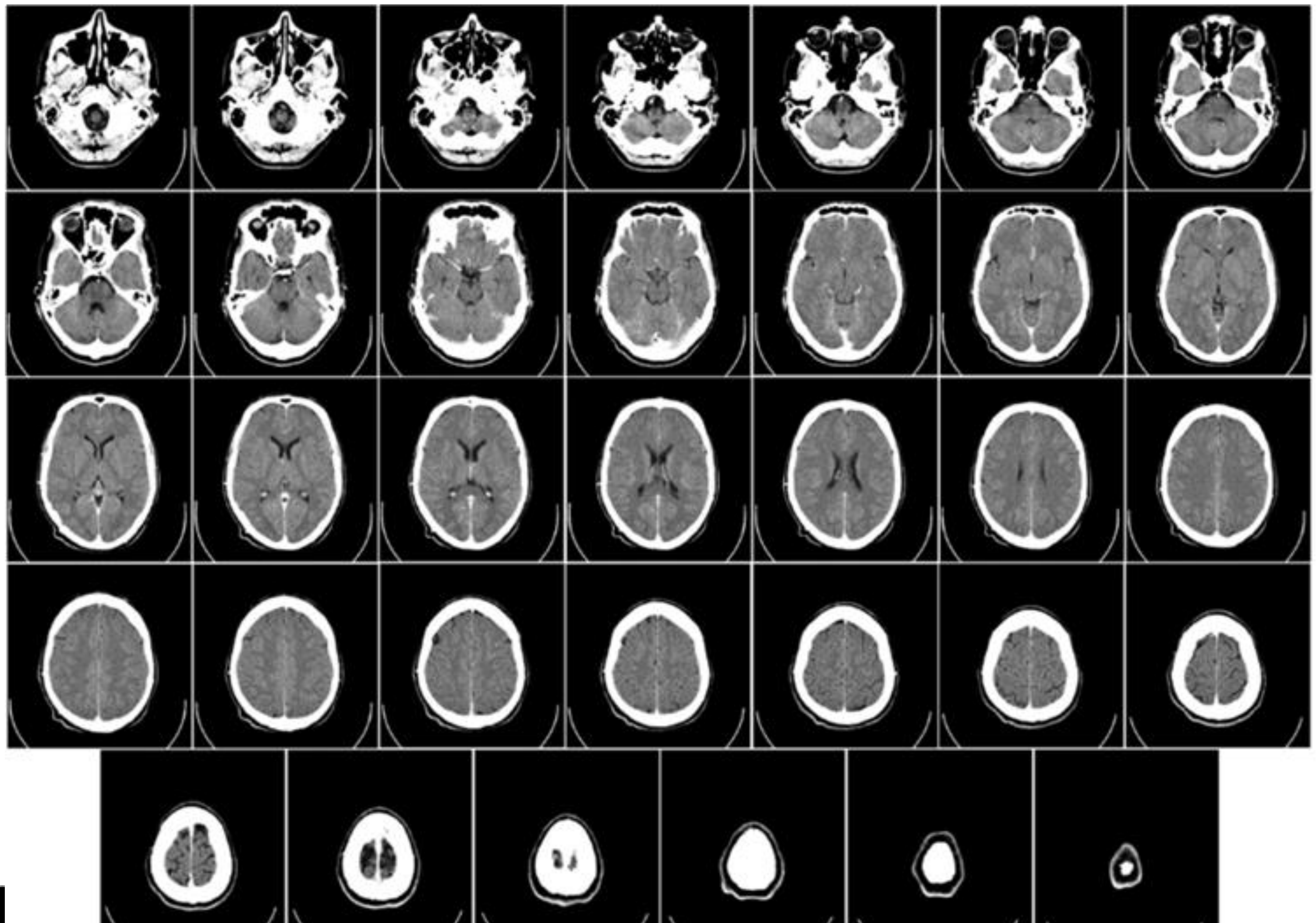


theta = 0:10:170

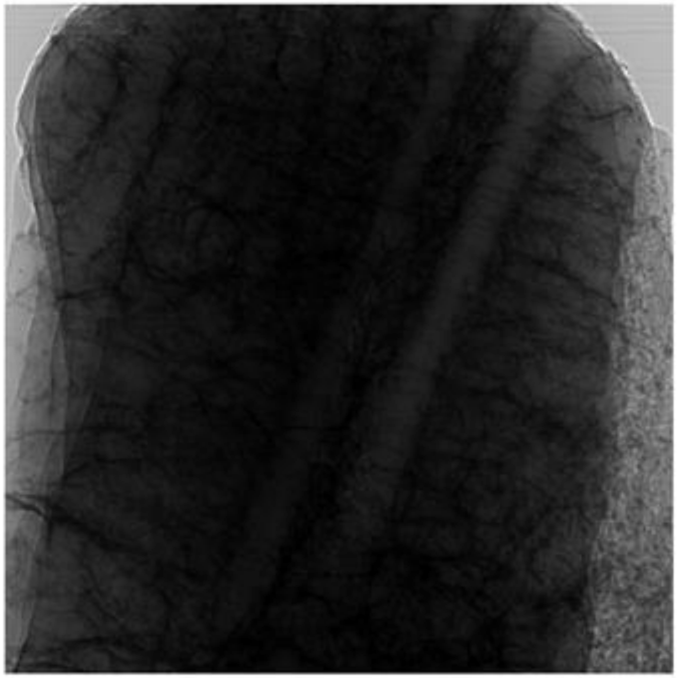


theta = 0:1:170

Medical applications: CT scanner

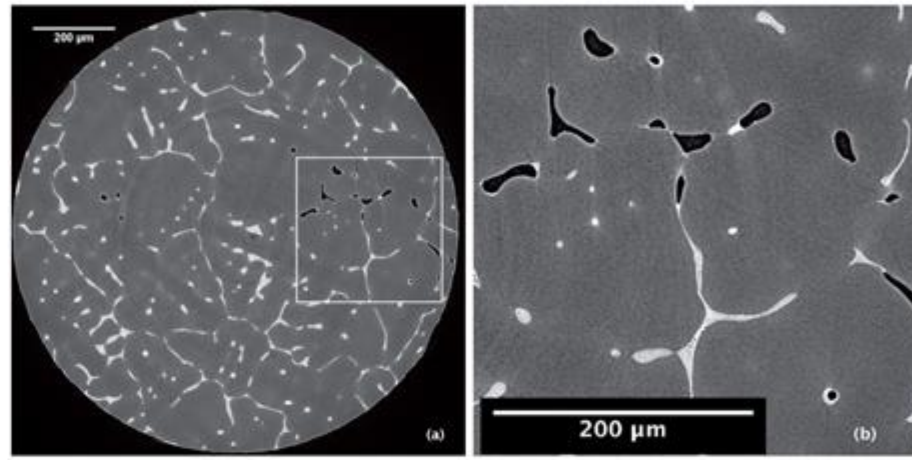


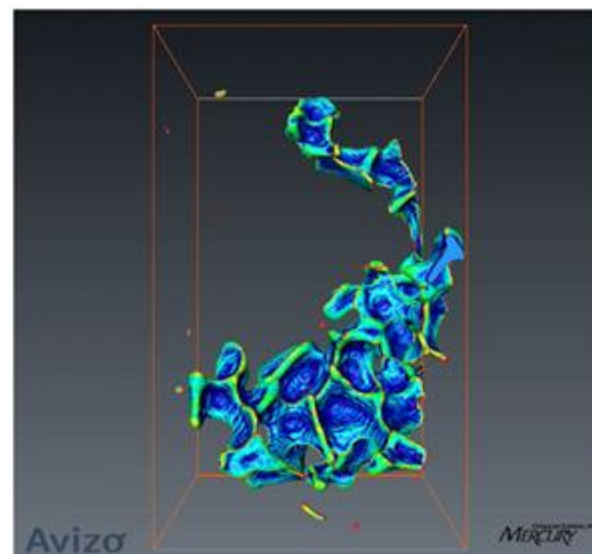
Tomography in material sciences



360degree X-ray tomography
Milan Felberbaum
STI-IMX-LSMX

Cylinder of an Al-Cu Alloy





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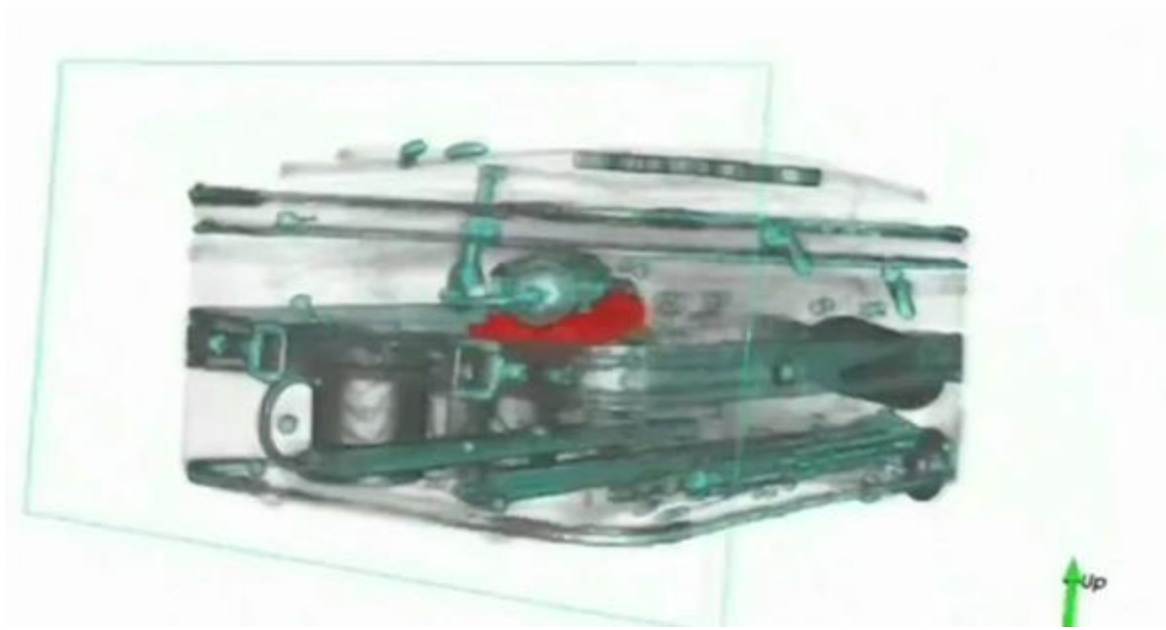
Tomography in everyday life

- Airport security – 3D Computed Tomography



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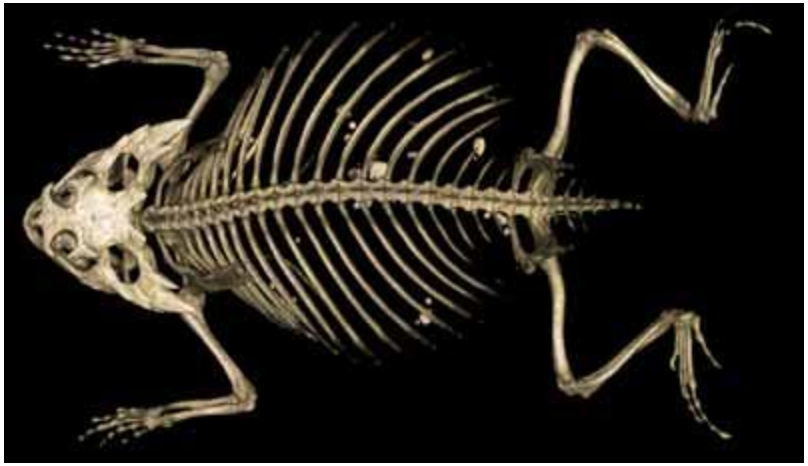
Tomography in biology



High-Resolution X-ray Computed Tomography

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DIGITAL MORPHOLOGY

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